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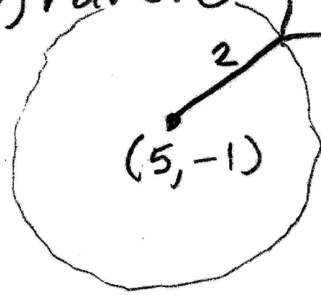
① $\operatorname{div}\langle P,Q\rangle = P_x + Q_y = 2x + x = \boxed{3x}$

$\operatorname{curl}\langle P,Q\rangle = Q_x - P_y = y - 2y = \boxed{-y}$

② $\oint_{\partial B} \langle P,Q\rangle \cdot \underbrace{\langle dy, -dx\rangle}_{\vec{N}\,ds} = \iint_B 3x\, dA = \int_0^2 \int_3^8 3x\, dy\, dx = \boxed{30}$

③ $\oint_{\partial D} \langle P,Q\rangle \cdot \underbrace{\langle dx, dy\rangle}_{\vec{T}\,ds} = \iint_D -y\, dA = \int_0^{2\pi}\int_0^2 -(-1+r\sin\theta)\, r\, dr\, d\theta = \boxed{4\pi}$

using shifted polar coordinates $x = 5 + r\cos\theta$, $y = -1 + r\sin\theta$



④ $\oint_{\partial B} \langle P,Q\rangle \cdot \langle dx, dy\rangle = \int_0^2 \int_3^8 -y\, dy\, dx = \boxed{-55}$

⑤ $\oint_{\partial D} \langle P,Q\rangle \cdot \langle dy, -dx\rangle = \int_0^{2\pi}\int_0^2 3(5+2\cos\theta)\, r\, dr\, d\theta = \boxed{60\pi}$

field	A	B	C	D	E	F	G	H	1
curl	0	0	+	+	−	0	−	0	2
div	+	−	0	−	0	−	+	0	

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③ $P = x^2 + yz$

$Q = xyz$

$R = 144(x+y+z)^{-1}$

$\operatorname{div}\vec{F} = P_x + Q_y + R_z = 2x + xz - 144(x+y+z)^{-2} = \boxed{-4}$ (@ (1, 2, 3))

$\operatorname{curl}\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$= \left\langle -\frac{144}{(x+y+z)^2} - xy,\ y + \frac{144}{(x+y+z)^2},\ yz - z \right\rangle = \boxed{\langle -11, 11, 3\rangle}$

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① $r-7=\sqrt{9}\cos\psi$
$z+1=\sqrt{9}\sin\psi$
$0\le\psi\le 2\pi$

$\Rightarrow$ $x=(7+3\cos\psi)\cos\theta$ $\quad 0\le\theta\le 2\pi$
$y=(7+3\cos\psi)\sin\theta$ $\quad 0\le\psi\le 2\pi$
$z=3\sin\psi-1$

$\vec r=\langle x,y,z\rangle$. $\quad \vec r_\psi=\langle -3\sin\psi\cos\theta,\ -3\sin\psi\sin\theta,\ 3\cos\psi\rangle$
$r_\theta=\langle -(7+3\cos\psi)\sin\theta,\ (7+3\cos\psi)\cos\theta,\ 0\rangle$

$$\vec r_\psi\times\vec r_\theta=\underbrace{(7+3\cos\psi)}_{\ge 7-3=4>0}(3)\langle -\cos\psi\cos\theta,\ -\cos\psi\sin\theta,\ -\sin\psi\overbrace{(\cos^2\theta+\sin^2\theta)}^{1}\rangle$$

$\text{magnitude}^2=\cos^2\psi(\cos^2\theta+\sin^2\theta)+\sin^2\psi=1$

$|\vec r_\psi\times\vec r_\theta|=(7+3\cos\psi)(3)$ $\quad \text{area}=\int_0^{2\pi}\int_0^{2\pi}(21+9\cos\psi)\,d\psi)\,d\theta$
$=21(2\pi)^2$

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② $x/2=\sqrt{1}\sin\varphi\cos\theta$
$y/3=\sqrt{1}\sin\varphi\sin\theta$
$z/4=\sqrt{1}\cos\varphi$

$\Rightarrow$ $\vec r=\langle 2\sin\varphi\cos\theta,\ 3\sin\varphi\sin\theta,\ 4\cos\varphi\rangle$
$\Rightarrow$ $\vec r_\varphi=\langle 2\cos\varphi\cos\theta,\ 3\cos\varphi\sin\theta,\ -4\sin\varphi\rangle$
$\vec r_\theta=\langle -2\sin\varphi\sin\theta,\ 3\sin\varphi\cos\theta,\ 0\rangle$

$0\le\varphi\le\pi,\ 0\le\theta\le 2\pi$

$\vec r_\varphi\times\vec r_\theta=\langle 12\sin^2\varphi\cos\theta,\ 8\sin^2\varphi\sin\theta,\ 6\cos\varphi\sin\varphi(\cos^2\theta+\sin\theta)\rangle$

$|\vec r_\varphi\times\vec r_\theta|=2\sin\varphi\sqrt{36\sin^2\varphi\cos^2\theta+16\sin^2\varphi\sin^2\theta+9\cos^2\varphi}$

$\int_0^{\pi}\left(\int_0^{2\pi}|\vec r_\varphi\times\vec r_\theta|\,d\theta\right)d\varphi\approx\boxed{111.546}$

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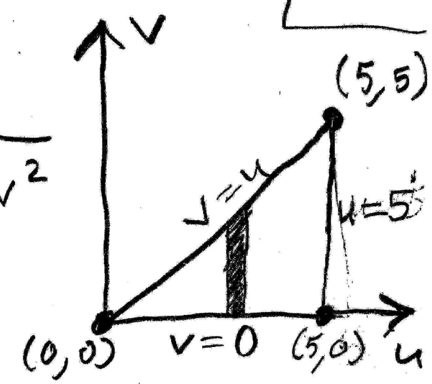
③ $\vec{r}_u = \langle 2u, 2u, v \rangle, \quad \vec{r}_v = \langle -2v, 2v, u \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u^2 - 2v^2, -2v^2 - 2u^2, 4uv + 4uv \rangle$

$|\vec{r}_u \times \vec{r}_v| = 2\sqrt{u^4 - 2u^2v^2 + v^4 + u^4 + 2u^2v^2 + v^4 + 16u^2v^2}$

$= 2\sqrt{2(u^4 + v^4 + 8u^2v^2)}$



$\text{area} = \int_0^5 \left(\int_0^u 2\sqrt{2(u^4 + 8u^2v^2 + v^4)}\, dv \right) du \approx \boxed{127.483}$

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① $\boxed{\langle x, y, z \rangle = \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle, \; 0 \le \theta \le \frac{\pi}{2}, \; 0 \le \varphi \le \frac{\pi}{2}}$

② $\vec{N}\, dA = \pm\, \vec{r}_\varphi \times \vec{r}_\theta \, d\varphi\, d\theta.$

$\vec{r}_\varphi = \langle \cos\varphi \cos\theta, \cos\varphi \sin\theta, -\sin\varphi \rangle$

$\vec{r}_\theta = \langle -\sin\varphi \sin\theta, \sin\varphi \cos\theta, 0 \rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = \sin\varphi \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle$

$= \frac{1}{\sqrt{2}} \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \rangle > 0 \; @ \left(\frac{\pi}{4}, \frac{\pi}{4}\right).$

$\vec{r}_\varphi \times \vec{r}_\theta = \langle +, +, + \rangle @ \left(\frac{\pi}{4}, \frac{\pi}{4}\right) \Leftarrow$

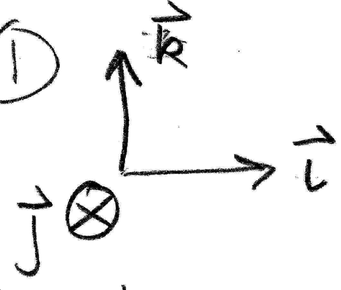
$\vec{N}\, dA = \vec{r}_\varphi \times \vec{r}_\theta \, d\varphi\, d\theta$

$= \boxed{(\sin\varphi) \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle \, d\varphi\, d\theta}$

③ $\vec{F} = \langle \cos\varphi + \sin\varphi \cos\theta, \sin\varphi(\cos\theta + \sin\theta), \sin\varphi \sin\theta + \cos\varphi \rangle$

$\vec{F} \cdot (\vec{r}_\varphi \times \vec{r}_\theta) = (\sin\varphi)(1 + \cos\varphi \sin\varphi (\cos\theta + \sin\theta) + \sin^2\varphi \cos\theta \sin\theta)$

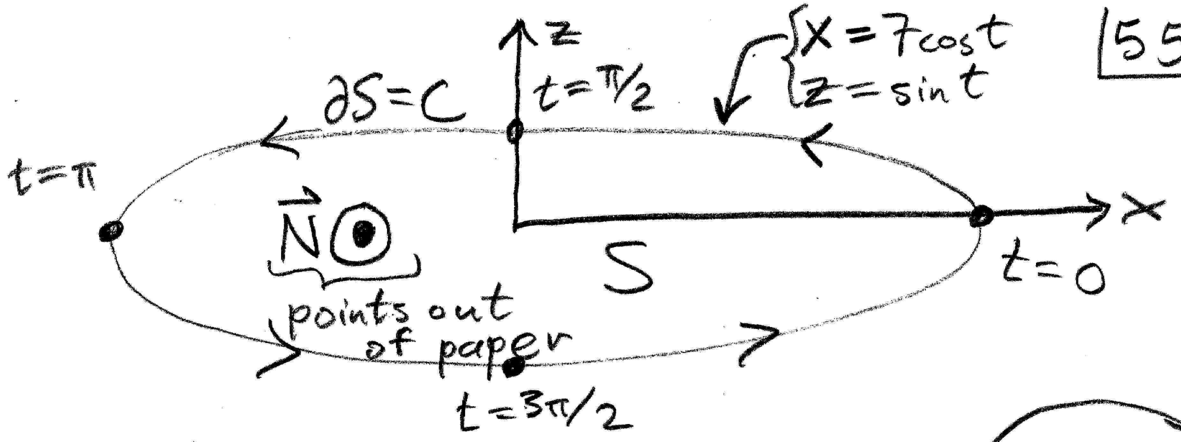
$\int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\vec{F} \cdot (\vec{r}_\varphi \times \vec{r}_\theta) \right) d\theta \right) d\varphi = \int_0^{\pi/2} (\sin\varphi) \left(\frac{\pi}{2} + \overbrace{2\cos\varphi \sin\varphi}^{\text{use } u = \sin\varphi} + \overbrace{\frac{1}{2} \underbrace{\sin^2\varphi}_{1-\cos^2\varphi}}^{\text{use } u = \cos\varphi} \right) d\varphi = \boxed{\frac{\pi}{2} + \frac{2}{3} + \frac{1}{3}}$

① 

$\vec{j}$ points into paper

right-handed coordinate system: $\vec{i}\times\vec{j}=\vec{k}$

hand × fingers = thumb



For S, use essentially polar coordinates with u, v instead of r, θ: $\boxed{\langle x,y,z\rangle=\langle 7u\cos v,\ 4,\ u\sin v\rangle}$ for $(u,v)\in[0,1]\times[0,2\pi]$.

Curl fingers like C; thumb points as $\vec{N}$. $\vec{N}=-\vec{j}$

$$\vec{N}\,dA=\pm\vec{r}_u\times\vec{r}_v\,du\,dv=\pm\langle 7\cos v,0,\sin v\rangle\times u\langle -7\sin v,0,\cos v\rangle\,du\,dv$$

$$=\pm u\langle 0,-7\underbrace{(\sin^2 v+\cos^2 v)}_{1},0\rangle\,du\,dv=\underbrace{\langle 0,-7u,0\rangle\,du\,dv}$$

has direction $-\vec{j}$ when $u>0$.

② $\alpha=\beta$ by Stokes' Thm where $\begin{cases}\alpha=\iint_S(\text{curl }\vec{F})\cdot\vec{N}\,dA\ \& \\ \beta=\oint_{\partial S}\vec{F}\cdot\vec{T}\,ds.\end{cases}$

$$\text{curl }\vec{F}=\langle -3z^2,-3x^2,-3y^2\rangle$$

$$=-3\langle u^2\sin^2 v,\ 7^2u^2\cos^2 v,\ 4^2\rangle\Rightarrow(\text{curl }\vec{F})\cdot\vec{N}\,dA=3\cdot 7^3u^3\cos^2 v\cdot du\,dv$$

$$\Rightarrow\alpha=3\cdot 7^3\int_0^1 u^3\,du\int_0^{2\pi}\cos^2 v\,dv=\boxed{7^3 3\pi/4}$$

$$\frac{3\pi}{4}=\int_0^{2\pi}\cos^4 t\,dt$$

$$\beta=\int_C(y^3dx+z^3dy+x^3dz)=\int_0^{2\pi}\left[(4^3)(-7\sin t)+(\sin^3 t)(0)+(7^3\cos^3 t)(\cos t)\right]dt=\boxed{\frac{7^3 3\pi}{4}}$$