Subsets of $\mathbb{R}^2$:

$\mathbb{R}^2$ is open & closed

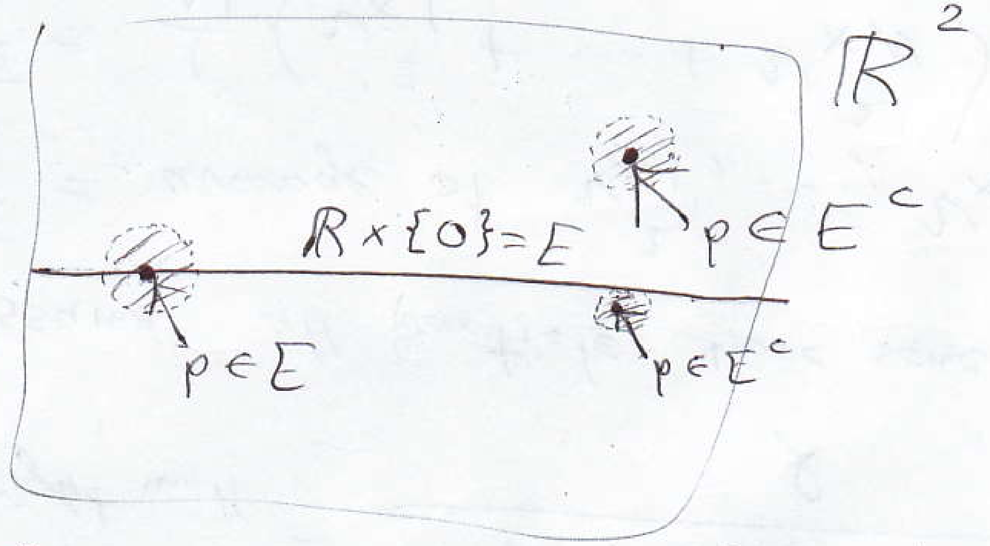
ϕ is open & closed

More examples : 2.21

$A \times B = \{(a,b) : a \in A \ \& \ b \in B\}$

$\mathbb{R} \times \{0\} = \{(x,y) : x \in \mathbb{R} \ \& \ y \in \{0\}\}$

$E = \mathbb{R} \times \{0\} = \{(x,0) : x \in \mathbb{R}\}$ closed, not open



$E \subset \mathbb{R}^2$ is closed if and only if

$\forall p \in E^c \ \exists r > 0 \ N_r(p) \subset E^c$

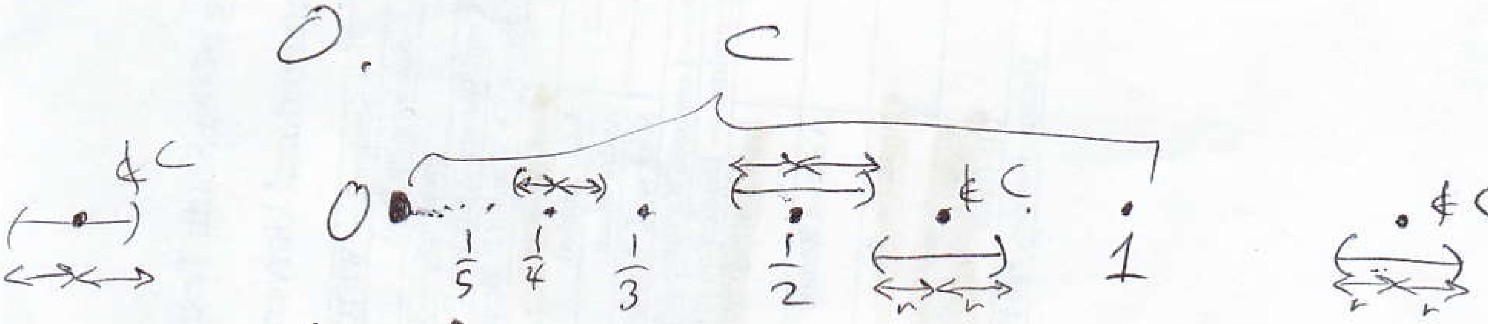
$E \subset \mathbb{R}^2$ is ~~closed~~ open if and only if

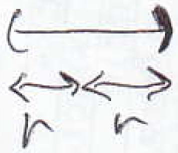
$\forall p \in E \ \exists r > 0 \ N_r(p) \subset E$.

Exercise #5, Ch. 2:

Construct a bounded set of real numbers with exactly 3 limit points.

$C = \{\frac{1}{n} : n \in \{1, 2, 3, \ldots\}\}$ has exactly 1 limit point in $\mathbb{R}$: 0.





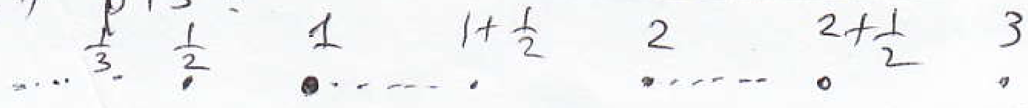
$\forall r > 0 \quad \underbrace{N_r(0)}_{(-r,r)} \cap C \neq \emptyset$

For example, if $0 < x < 1$, then let $a = \max\{\frac{1}{n} : \frac{1}{n} < x\}$ & $b = \min\{\frac{1}{n} : \frac{1}{n} > x\}$

Let $r = \min\{|x-a|, |x-b|\}$

Then $N_r(x) \cap C \subseteq \{x\}$

3 limit pts:



disk = 2D ~~ball~~ ball

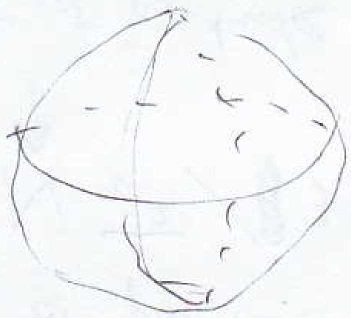
N for neighborhood

open ball: $N_r(p) = \{q \in X : d(p,q) < r\}$

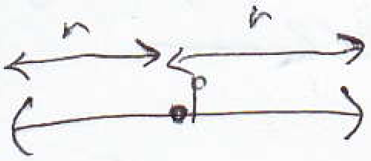
closed ball: $\cancel{N_r(p)}$ $\{q \in X : d(p,q) \le r\}$

X can be any metric space

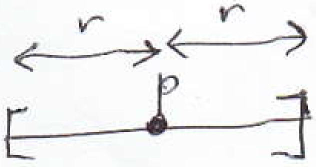
In $\mathbb{R}^3$:



In $\mathbb{R}$



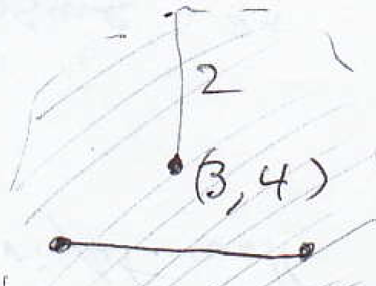
open



closed

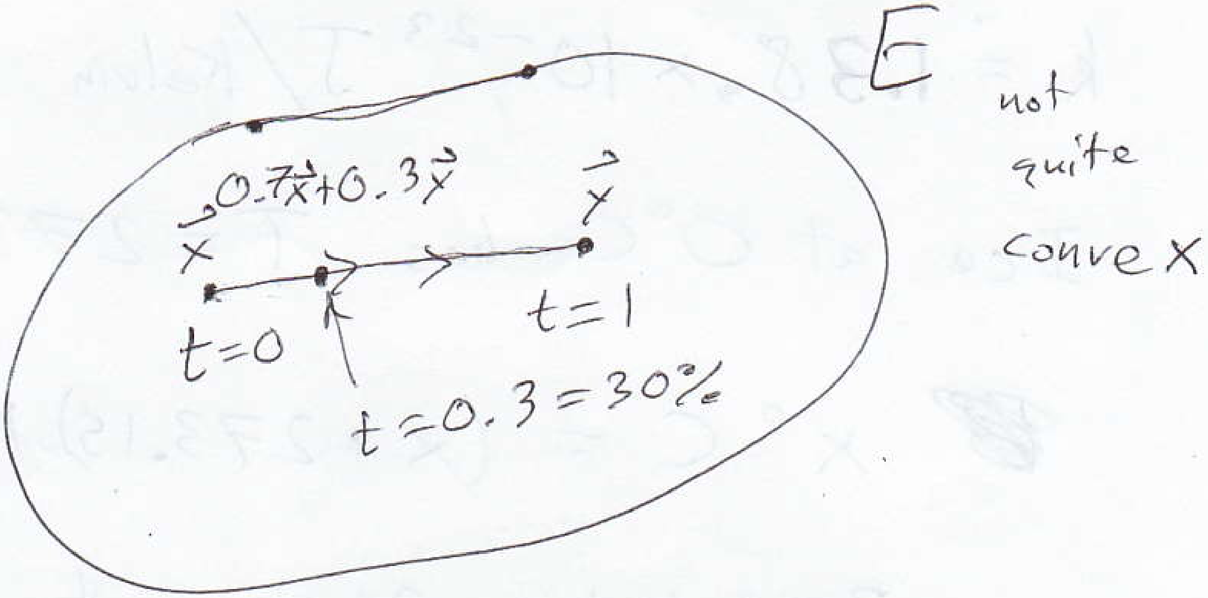
$\{(x,y) : (x-3)^2 + (y-4)^2 < 2^2\}$

is convex.



$\{(x,y) :$
$(x-3)^2 + (y-4)^2 = 2^2\}$

not convex.



$E \subset \mathbb{R}^k$ is called convex if

$$\left[\begin{array}{l} \forall \vec{x}, \vec{y} \in E \quad \forall t \in (0,1) \\ (1-t)\vec{x} + t\vec{y} \in E \end{array}\right]$$

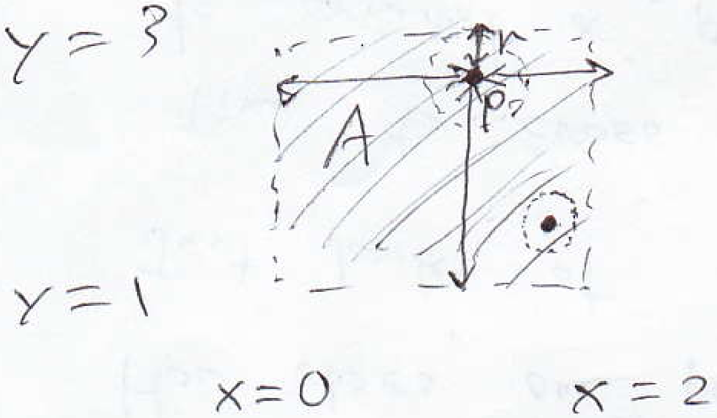
Same as $\left[\begin{array}{l} \forall \vec{x}, \vec{y} \in E \ \forall \lambda \in [0,1] \\ \lambda \vec{x} + (1-\lambda)\vec{y} \in E \end{array}\right]$

Same as: $\forall \vec{x}, \vec{y} \in E$ the line segment between $\vec{x}$ & $\vec{y}$ is contained in E.

$A = (0,2)\times(1,3)$

$= \{(x,y) : 0 < x < 2 \ \& \ 1 < y < 3\}$

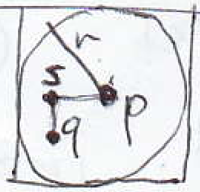
A is an open subset of $\mathbb{R}^2$.



Proof: Suppose $p = (x,y) \in A$. Let r be the minimum of the distances from p to the four edges, that is,

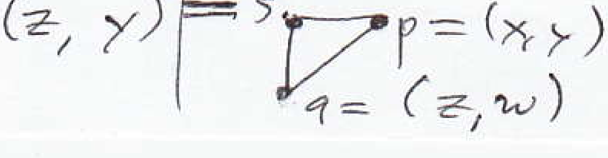
$|x-0|, \ |x-2|, \ |y-3|, \ |y-1|.$

Claim: $N_r(p) \subset A$. To see this, suppose $q = (z,w) \in N_r(p)$. We need to show that $(z,w) \in A$.



$r > d(p,q) = \underbrace{\sqrt{(z-x)^2 + (w-y)^2}}$

$r > d(p,q) \geq \underbrace{d(s,p)}_{\underbrace{|z-x|}_{\sqrt{(z-x)^2+0}}}, \ \overset{\geq}{\underbrace{d(s,q)}_{|w-y|}}$



$$0 < x < 2 \qquad 0 \le x - r < x \le x + r \le 2$$

$$1 \cancel{0} < y < 3 \qquad 1 \le y - r < y \le y + r \le 3$$

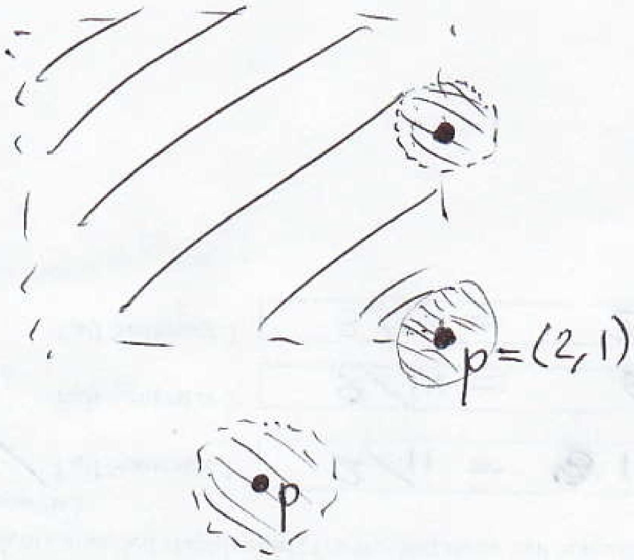
because of our choice of r.

$$r > |z - x| \iff x - r < z < x + r$$

$$r > |w - y| \iff y - r < w < y + r$$

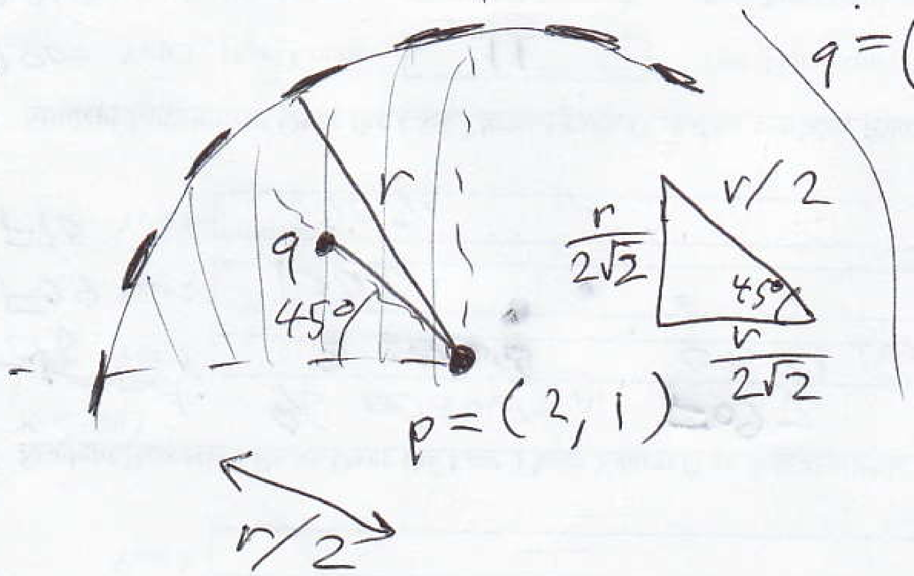
$$\checkmark \quad q = (z, w) \in A \iff \begin{cases} 0 < z < 2 \\ 1 \cancel{0} < w < 3 \end{cases}$$

$A = (0,2) \times (1,3)$



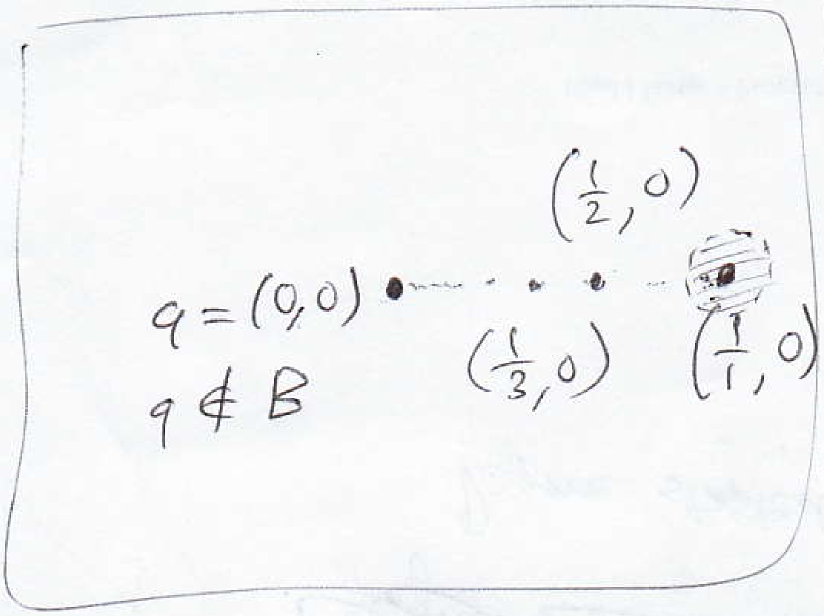
Is A a closed subset of $\mathbb{R}^2$?

No. If p is, for example, $(2,1)$, then, for any $r>0$, $q = \left(2 - \frac{r}{2\sqrt{2}}, 1 + \frac{r}{2\sqrt{2}}\right)$ is in $N_r(p)$ and in A. So, $p \in A^c$ but $\forall r>0$ $N_r(p) \not\subset A^c$.



$(0,2)\times(1,3)$ is open but not closed.

$\mathbb{R}^2$



$B = \{(\frac{1}{n}, 0) : n = 1, 2, 3, \ldots\}$

B is not open nor closed as as a subset of $\mathbb{R}^2$

Not open: $p = (1,0) \in B$

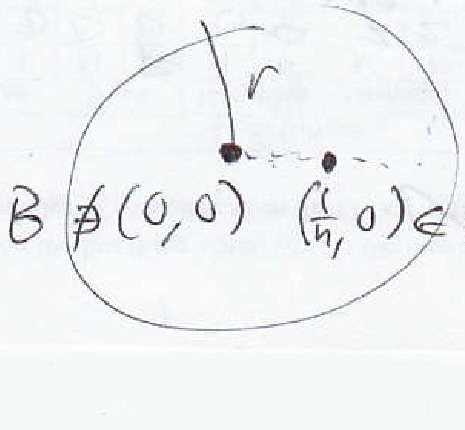
$\forall r > 0$, ~~if~~ let $s = \min(\frac{r}{2}, \frac{1}{3})$.

Then $(1-s, 0) \in N_r(p)$, but $(1-s,0) \notin A$, so $N_r(p) \not\subset B$.

Not closed: $q = (0,0) \in B^c$

but $\forall r > 0 \ \exists n \in \{1, 2, 3, \ldots\}$

$0 < \frac{1}{n} < r$, so $(\frac{1}{n}, 0) \in B$ & $(\frac{1}{n}, 0) \in N_r(q)$



$D = \{ m + \frac{1}{n} : m = 0, 1, 2 \ \& \ n = 1, 2, 3, \dots \}$

D has exactly 3 limit pts:

0, 1, 2.