"Soft" proof that $f(x) = \sqrt[3]{x}$ is uniformly continuous on $[-1, 1]$:

- $[-1, 1]$ is compact & f is continuous
- continuity implies uniform continuity on compact sets.
- QED

"Hard" proof that $f(x) = \sqrt[3]{x}$ is uniformly continuous on $[-1, 1]$:

Given $\varepsilon > 0$, ~~such that~~ we need to find δ such that $\forall x, y \in [-1, 1]$ $(|x-y| < \delta \Rightarrow |x^{1/3} - y^{1/3}| < \varepsilon)$.

~~If $x, y \in [-\varepsilon^3, \varepsilon^3]$, then $|x^{1/3} - y^{1/3}|$~~
~~$\leq |x^{1/3}| + |-y^{1/3}| \leq \sqrt[3]{\varepsilon^3} + \sqrt[3]{\varepsilon^3} = 2\varepsilon^{1/3}$~~

If $x, y \in (-(\varepsilon/2)^3, (\varepsilon/2)^3)$, then

$|x^{1/3} - y^{1/3}| \leq |x^{1/3}| + |-y^{1/3}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

If $x, y \in [-1, 0)$ or $x, y \in (0, 1]$, then

$|x^{1/3} - y^{1/3}| = \frac{1}{3}|z^{-2/3}||x-y|$ for some z between x & y, by the mean value theorem.

If $x, y \in [-1, -\varepsilon^3/16]$ or $x, y \in [\varepsilon^3/16, 1]$,

then $|z| \geq \varepsilon^3/16$, so $|z^{-2/3}| \leq \varepsilon^{-2} 16^{-2/3}$

So, if $|x-y| < \varepsilon^3$, then

$$|x^{\frac{1}{3}}-y^{\frac{1}{3}}| \le \frac{1}{3} z^{-2/3}|x-y| < \frac{1}{3}\varepsilon^{-2}\frac{1}{\sqrt[3]{256}}\varepsilon^3 < \varepsilon.$$

Let $\boxed{\delta = \varepsilon^3/16} = (\varepsilon/2)^3/2.$

If $x, y \in [-1, 1]$ & $|x-y| < \delta$,
then (I) $x, y \in [-1, -\varepsilon^3/16]$ or
(II) $x, y \in (-(\varepsilon/2)^3, (\varepsilon/2)^3)$ or
(III) $x, y \in [\varepsilon^3/16, 1]$;
in case II, $|x^{1/3}-y^{1/3}| \le \varepsilon$;
in cases I & III, $|x-y| < \delta < \varepsilon^3$, so,
as argued above, $|x^{1/3}-y^{1/3}| < \varepsilon$. QED

The hard proof takes longer, but gives additional information: an explicit δ that works.