4335 Final Exam (open-note; closed-book)

★ Name: ______________

1) True / False: [20 points]

a) ______: If $E \subset \mathbb{R}$ is bounded and $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in E, then $(x_n)_{n=1}^{\infty}$ converges to a point in E.

b) ______: If $E \subset \mathbb{R}$ is closed and $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in E, then $(x_n)_{n=1}^{\infty}$ converges to a point in E.

c) ______: If $f_n : [a,b] \to \mathbb{R}$ & $\int_a^b f_n(x)dx$ exists for all n, & $(f_n)_{n=1}^{\infty}$ converges uniformly to $f : [a,b] \to \mathbb{R}$, then $\left(\int_a^b f_n(x)dx\right)_{n=1}^{\infty}$ converges to $\int_a^b f(x)dx$.

d) ______: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ absolutely converges.

[25 points]

2) Prove that $\displaystyle\int_0^1 \frac{dx}{1+x^3} = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \cdots$

3) Give an example of a sequence of functions that converges pointwise but not uniformly. [20 points]

4. [15 points] Given $f, \alpha : [0,3] \to \mathbb{R}$

and $\alpha(x) = \begin{cases} 7 : & 2 < x \le 3 \\ 3 : & 1 \le x \le 2 \\ -2 : & 0 \le x < 1 \end{cases}$

and $f(x) = x^2$, compute $\int_0^3 f \, d\alpha$

5. [20 pts.] Prove that the 99th derivative of $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^n}$ exists at every point on the real line.