MATH 4335 FINAL EXAM

Name: ______________________________

Testing conditions:

- notes, books, and calculators are allowed;
- inter-student communication, telecommunication, and internet access are not allowed.

Date: Dec. 9, 2013.

Exercise	Point Possible	Score
1	34	
2	33	
3	33	
Total	100	

1. [34 points] Assuming that $a < b$ and f is convex on $[a, b]$, prove that $I = \int\limits_a^b f(x)\,dx$ exists and that $I \le \frac{1}{2}(f(a) + f(b))$.

2. [33 points] Prove that the cube root function $f(x) = \sqrt[3]{x}$ is uniformly continuous on $(-\infty, \infty)$. Hint: Separately prove f that is uniformly continuous on each of $(-\infty, -1]$, $[-1, 1]$, and $[1, \infty)$.

3. [33 points] Suppose that $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ are powers series such that $\sqrt[n]{|a_n|} \to 1/2$ and $\sqrt[n]{|b_n|} \to 1/3$. Prove that the "product" series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $\geq 6/5$ (where $c_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \cdots + a_0 b_n$).

You may assume that $\sqrt[n]{|x+y|} \leq \sqrt[n]{|x|} + \sqrt[n]{|y|}$ for all $x, y \in \mathbb{R}$ and $n > 0$, and that $\sqrt[n]{n} \to 1$.

Suggested proof outline:

(1) Show that $\{a_n\}$ and $\{b_n\}$ are bounded.

(2) Expresing c_n as $p_n + q_n$ where

$$p_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \cdots + a_m b_{n-m};$$
$$q_n = a_{m-1} b_{n-(m-1)} + a_{m-2} b_{n-(m-2)} + a_{m-1} b_{n-(m-3)} + \cdots + a_0 b_n$$

where m is $n/2$ rounded down, show that $\limsup \sqrt[n]{p_n} \leq 1/2$ and $\limsup \sqrt[n]{q_n} \leq 1/3$

(3) Deduce that $\sum_{k=0}^{\infty} c_k x^k$ has radius of convergence $\geq (1/2 + 1/3)^{-1}$.

(Extra credit: By choosing m differently, show that $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence ≥ 2.)