2.7 #2(e) Does

$$1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$$

converge? No:

$$s_{2n} = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} \cdots + \frac{1}{2n-1} - \frac{1}{(2n)^2}$$

$$= \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right)$$

$$- \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2n)^2}\right)$$

$$\geq \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

$$- \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots \frac{1}{n^2}\right)$$

$$\geq \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \frac{1}{4} \cdot 2$$

because $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2$.

Because $\left(\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n}\right)$ is not bounded above, neither is $s_1, s_2, s_3, \cdots$

2.7 #4 (a) $\sum_{n=1}^{\infty} \frac{1}{n}$ & $\sum_{n=1}^{\infty} \frac{1}{n}$

diverge but $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$ converges.

(b) Let $x_n = (-1)^{n+1}/n$ and

$y_n = 1$ if n odd else 0.

Then $\sum_{n=1}^{\infty} x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

converges (by the alternating series test) and $(y_n) = 1, 0, 1, 0, 1, 0, \ldots$

is bounded but

$\sum_{n=1}^{\infty} x_n y_n = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + 0 + \frac{1}{7} + \cdots$

diverges.

(c) Impossible. If $\sum x_n$ & $\sum (x_n + y_n)$

converge, say, to A & B, respectively,

then $\sum y_n = \sum ((x_n + y_n) - x_n)$ converges

to $B - A$.

2.7#4(d) Let $(x_n) = 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots$

Then $x_n = 0$ or $1/n$ for all n; so, $0 \le x_n \le 1/n$ for all n. Yet, $\sum_{n=1}^{\infty} (-1)^n x_n = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + 1/6 + 0 + 1/8 + \cdots$ diverges.

2.7#7(a) Assume $na_n \to \ell \ne 0$ & $0 < a_n$.

Then $\ell \ge 0$ because $na_n \ge 0$. Hence, $\ell > 0$; hence, $n \cdot a_n \ge \ell/2$ eventually, say, for all $n \ge N$. Hence, for all $n \ge N$,

$$a_1 + \cdots + a_n \ge a_1 + \cdots + a_{N-1} + \frac{\ell}{2N} + \cdots + \frac{\ell}{2n}.$$

$$= \frac{\ell}{2}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) + c$$

for some $c \in \mathbb{R}$ that depends on N and $a_1, \ldots, a_{N-1}$ but not on n.

(Precisely, $c = a_1 + \cdots + a_{N-1} - \frac{\ell}{2}\left(\frac{1}{1} + \cdots + \frac{1}{N-1}\right)$.)

Since $\frac{1}{1}, \frac{1}{1} + \frac{1}{2}, \frac{1}{1} + \frac{1}{2} + \frac{1}{3}, \ldots$ is not bounded above, $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots$ is not a bounded sequence.

Hence, $(a_1 + \cdots + a_n)$ diverges. ■

2.7 #7(b) Assume $a_n > 0$ & $n^2 a_n \to L$.

Then $n^2 a_n < L+1$ eventually. Therefore, $0 \le a_n \le (L+1)/n^2$ eventually. Since $\sum (L+1)/n^2$ converges, so does $\sum a_n$ (by the comparison test). ■

Shorter and better solution to 7(a):

Given $a_n > 0$ & $n a_n \to \ell \ne 0$, we have $\ell \ge 0$, and, hence, $\ell > 0$. Therefore, $\ell > \ell/2 > 0$; hence, $n a_n > \ell/2$ eventually; hence $0 \le \ell/(2n) \le a_n$ eventually. By the comparison test, since $\sum \ell/(2n)$ diverges, so does $\sum a_n$. (We know that $\sum \ell/(2n)$ diverges because if it converged to A, then $\sum 1/n$, which we know diverges, would converge to $(2/\ell)A$.) ■