Section 4.3 Exercises #3, 4, 2

#3b: Given $f$ continuous (cts.) at $c$, & $g$ cts. @ $f(c)$, we claim $g \circ f$ is cts. @ $c$.

Proof using sequences:

If $(x_n) \to c$ and $g(f(x_n))$ is defined for all $n \in \mathbb{N}$, then we need to show $(g(f(x_n))) \to g(f(c))$.
By continuity of $f$, $(f(x_n)) \to f(c)$.
By continuity of $g$, $(g(f(x_n))) \to g(f(c))$. ∎

Note how much easier that was than the $\varepsilon-\delta$ proof (3a) asked for:

3 a: Given $\varepsilon_1 > 0$, we must find $\delta_1 > 0$ such that

(1.1) $(|x-c| < \delta_1$ & $g(f(x))$ defined$) \Rightarrow$

(1.2) $$|g(f(x)) - g(f(c))| < \varepsilon_1 .$$

Let $\varepsilon_2 = \varepsilon_1$. By continuity of $g$, there is $\delta_2 > 0$ such that

(2.1) $(|y - f(c)| < \delta_2$ & $g(y)$ defined$) \Rightarrow$

(2.2) $$|g(y) - g(f(c))| < \varepsilon_2 = \varepsilon_1 .$$

By continuity of $f$, there is, given $\varepsilon_3 = \delta_2$, some $\delta_3 > 0$ with

(3.1) $(|x-c| < \delta_3$ & $f(x)$ defined$) \Rightarrow$

(3.2) $$|f(x) - f(c)| < \varepsilon_3 = \delta_2 .$$

Let $\delta_1 = \delta_3$. Then, given $x$ as in (1.1), we see that $x$ satisfies (3.1); hence, $x$ satisfies (3.2). Letting $y = f(x)$, we then use (1.1) & (3.2) to see that $y$ satisfies (2.1); hence, $y$ satisfies (2.2); hence, $x$ satisfies (1.2).

Thus, (1.1) $\Rightarrow$ (1.2) as needed. ∎

## Example:

#4a: Let $p=q=r=0$ & $f(x)=0$,

and $g(x) = (x \text{ if } x \neq 0 \text{ else } 1)$.

Then $g(f(x)) = 1$ (for all $x$).

Therefore, $\lim\limits_{x\to p} f(x) = \lim\limits_{x\to 0} 0 = 0 = q$ and

$\lim\limits_{x\to q} g(x) = \lim\limits_{x\to 0} x = 0 = r$ but

$\lim\limits_{x\to p} f(g(x)) = \lim\limits_{x\to 0} 1 = 1 \neq 0 = r$.

4b: If $f$ & $g$ are cts. & $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathbb{R}$, then $g \circ f$ is cts., by Thm. 4.3.9, and $\mathrm{dom}(g\circ f) = \mathbb{R}$. Given these three continuities, if $\lim\limits_{x\to p} f(x) = q$ & $\lim\limits_{x\to q} g(x) = r$, then $q = f(p)$ & $r = g(q)$ and, hence,

$r = g(f(p)) = \lim\limits_{x\to p} g(f(x))$.

4c: Our 4a example still works if we require that $f$ is continuous, but do not require $g$ continuous. Our example fails if we require $g$ to be continuous. Moreover, no other example for 4a can have

$g$ continuous because if we assume $\lim_{x\to p} f(x)=q$, $\lim_{x\to q} g(x)=r$, and $g$ cts. @ $q$,

then, given $p \neq x_n \in dom(g\circ f)$ for all $n\in\mathbb{N}$ & $(x_n)\to p$, we deduce

(1) $f(x_n)\longrightarrow q$, because $\lim_{x\to p} f(x)=q$,

and then (2) $g(f(x_n))\longrightarrow g(q)$, because of (1) & continuity of $g$,

and finally (3) $g(q)=r$, because of continuity of $g$ & $\lim_{x\to q} g(x)=r$.

Thus, our assumptions imply $\lim_{x\to p} g(f(x))=r$ using (2) & (3) & the sequential characterization of continuity.

#2a: $f(x) = \mathbf{-37.2}$ (or any other constant) defines a "onetinuous" function on $\mathbb{R}$.

2b: $f(x) = x$ defines an "equaltinuous" function on all of $\mathbb{R}$. Moreover, $f$ is nowhere "onetinuous" because, given $c \in \mathbb{R}$ and $\varepsilon = 1/2$, $|x-c| < \delta = 1$ does not imply $|f(x) - f(c)| = |x-c| < \varepsilon = 1/2$! ($x = c + 3/4$ is a counterexample.) (The "because..." is not required.)

2c: $f(x) = 2x$ is "lesstinuous" on $\mathbb{R}$. (Use $\delta = \varepsilon/2 < \varepsilon$.) But $f$ is nowhere "equaltinuous" because, given $c \in \mathbb{R}$ & $\varepsilon = 1$, $|x-c| < \delta = \varepsilon = 1$ does not imply $|f(x) - f(c)| = 2|x-c| < \varepsilon = 1$. ($x = c + 3/4$ is a counterexample.)

2d: Every "lesstinuous" function $f$ is continuous because, given $c \in \mathrm{dom}(f)$ & $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$ as in the "lesstinuous" def., $\delta$ is also as required by the def. of continuity simply because if $0 < \delta < \varepsilon$ then $0 < \delta$. Conversely, given merely $0 < \delta_1$ & the implication $(|x-c| < \delta_1 \text{ \& } f(x) \text{ defined}) \Rightarrow |f(x) - f(c)| < \varepsilon$, we can let $\delta_2 = \min\{\delta_1, \varepsilon/2\}$ to obtain $0 < \delta_2 < \varepsilon$ and

$$|x-c| < \delta_2 \Rightarrow |x-c| < \delta_1 \Rightarrow |f(x) - f(c)| < \varepsilon$$

for all $x$ where $f(x)$ is defined.

Thus, every continuous function is "lesstinuous."