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Topology

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- Context: the real line $\mathbb{R}, <$

A	$\min(A)$	$\max(A)$	L	$\inf(A)$ $\max(L)$	U	$\sup(A)$ $\min(U)$
$[3,4)$	3	DNE	$(-\infty,3]$	3	$[4,\infty)$	4
$(3,4)$	DNE	DNE	$(-\infty,3]$	3	$[4,\infty)$	4
$(3,4]$	DNE	4	$(-\infty,3]$	3	$[4,\infty)$	4
$[3,4]$	3	4	$(-\infty,3]$	3	$[4,\infty)$	4
$(-\infty,5)$	DNE	DNE	$\emptyset$	DNE	$[5,\infty)$	5
$(-\infty,5]$	DNE	5	$\emptyset$	DNE	$[5,\infty)$	5

$L = \{x \in \mathbb{R} \mid x \leq a \text{ for all } a \in A\}$
$U = \{x \in \mathbb{R} \mid a \leq x \text{ for all } a \in A\}$
$\text{glb}(A) = \inf(A) = \max(L)$
$\text{lub}(A) = \sup(A) = \min(U)$
DNE = "Does not Exist"

A	$\min(A)$	$\max(A)$	L	$\max(L)$	U	$\min(U)$
$\emptyset$	DNE	DNE	$(-\infty,\infty)$	DNE	$(-\infty,\infty)$	DNE
$\mathbb{R}$	DNE	DNE	$\emptyset$	DNE	$\emptyset$	DNE

A <u>binary operation</u> on a set A is a function $f: A \times A \to A$
We often write $f(a,b)$ as afb.

We will assume there is a set $\mathbb{R}$ with element $0, 1$ and binary operations $+, \cdot$ such that:

- $(x+y)+z = x+(y+z)$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ } <u>Associativity</u> $\to$ $\underbrace{x \cdot (y+z) = (x \cdot y) + (x \cdot z)}_{\text{Distributivity}}$
- $x+y = y+x$
- $x \cdot y = y \cdot x$ } <u>Commutativity</u>
- $x+0 = x$
- $x \cdot 1 = x$ } <u>Identity</u>
- for all x, there exists $(-x)$ such that $x+(-x)=0$
- for all $x \neq 0$, there exists $\left(\frac{1}{x}\right)$ such that $x \cdot \left(\frac{1}{x}\right) = 1$

From these assumption, define

$x-y = x+(-y)$

$x/y = x\cdot\left(\frac{1}{y}\right)$ if $y \neq 0$

We also assume $\mathbb{R}$ has an order relation $<$ such that:

<u>compatibility</u> $\begin{cases} x<y \Rightarrow x+z<y+z \\ (x<y \text{ and } 0<z) \Rightarrow x\cdot z < y\cdot z \end{cases}$

(least upper bound property)

$\emptyset \neq A \subset \mathbb{R}$ and A has an upper bound $\Rightarrow$ A has a least upper boun

(density) $x<y \Rightarrow x<z<y$ for some z

$(0\neq 1$ & $\underbrace{\text{other axioms}}_{\text{all axioms but density}}$ & $x<y) \Rightarrow x < \frac{x+y}{2} < y$

- A set $A \subset \mathbb{R}$ is called inductive if $1 \in A$ and $x \in A \Rightarrow x+1 \in A$
- $[1,\infty)$ is inductive.
- $\{1\} \cup [2,\infty)$ is inductive
- $\{1,2\} \cup [3,\infty)$ is inductive.
- $\{1,\dots,n\} \cup [n+1,\infty)$ is inductive
- $\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \dots\}$ is inductive

$\mathbb{Z}_+ = \bigcap \{A \subset \mathbb{R} \mid A \text{ is inductive}\}$

Our official definition

<u>Claim</u> $\mathbb{Z}_+$ is an inductive set.

$1 \in \mathbb{Z}_+$

Suppose $n \in \mathbb{Z}_+$. We need to show $n+1 \in \mathbb{Z}_+$

Suppose $A \subset \mathbb{R}$ is inductive. Since $n \in \mathbb{Z}_+$, $n \in A$. Since A is inductive, $n \in A \Rightarrow n+1 \in A$. So, $n+1 \in A$. The above argument works for any inductive A, so $n+1 \in \mathbb{Z}_+$. $\square$

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Principle of Induction

If $B \subset \mathbb{Z}_+$ and B is inductive, then $B = \mathbb{Z}_+$

Proof By definition of $\mathbb{Z}_+$, $\mathbb{Z}_+ \subset B$. But then $B \subset \mathbb{Z}_+ \subset B$; so, $B = \mathbb{Z}_+$. □

Proofs by induction:

You want to prove every $n \in \mathbb{Z}_+$ has property P.

1.) $B = \{n \in \mathbb{Z}_+ \mid n \text{ has } P\}$

2) Prove $1 \in B$ (that is, 1 has P).

3) Prove $\underbrace{n \in B}_{n \text{ has } P} \Rightarrow n+1 \in B$

Every $n \in \mathbb{Z}_+$ is even or odd.

- 1 is odd
- If n is even, then n+1 is odd (Why? $n = 2m \Rightarrow n+1 = 2m+1$)
- If n is odd, then n+1 is even (Why? $n = 2m+1 \Rightarrow n+1 = 2m+2 = 2(m+1)$)
- If n is even or odd, then n+1 is even or odd.

Strong Induction Principle

If $B \subset \mathbb{Z}_+$ and $S_n \subset B \Rightarrow n \in B$, then $B = \mathbb{Z}_+$.

$S_1 = \emptyset$ $S_3 = \{1,2\}$

$S_2 = \{1\}$ $S_{n+1} = \{1, \dots, n\}$ $S_{n+1} = [1,n] \cap \mathbb{Z}_+$

Well-ordered Principle

$\emptyset \neq A \subset \mathbb{Z}_+ \Rightarrow \min(A)$ exists