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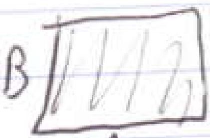
Today: §5 arbitrary products
§6 finite sets

$A\times B=\{(a,b)\mid a\in A \text{ \& } b\in B\}$ = B  A

$A\times B\times C=\{(a,b,c)\mid a\in A \text{ \& } b\in B \text{ \& } c\in C\}$ = B  A

π product
$n! := \prod_{i=1}^{n} \cdot 2\cdot 3\cdot\ldots\cdot n$

Ordered m-tuple: $(a_1, a_2, a_3, \ldots, a_{m-2}, a_{m-1}, a_m)$
is a function with domain $\{1,\ldots,m\}$

$i \mapsto a_i$ ith coordinate.

$A_1, A_2, \ldots, A_m$ are sets, then $\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \ldots$
$= \{f: m \to \bigcup_{i=1}^{m} A_i \mid \text{for all } i\in\{1,\ldots,m\}\ f(i)\in A_i\}$

$= \{(a_1, a_2, \ldots, a_m \in A_m\}$

An ordered ω-tuple is a function with domain
$\underbrace{\mathbb{Z}_+}_{\{1,2,3,4,\ldots\}}$

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Notation: $\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right) = \left(\frac{1}{2^{n-1}}\right)_{n \in \mathbb{Z}_+}$

$$\begin{array}{cccc} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ \| & \| & \| & \| \\ \frac{1}{2^{1-1}} & \frac{1}{2^{2-1}} & \frac{1}{2^{3-1}} & \frac{1}{2^{4-1}} \end{array}$$

These are also called sequences.

$$\prod_{i=1}^{\infty} A_i = \prod_{i \in \mathbb{Z}_+} A_i = A_1 \times A_2 \times A_3 \times \ldots$$

$$= \{(a_1, a_2, a_3, \ldots) \mid a_1 \in A_1 \ \& \ a_2 \in A_2 \ \& \ a_3 \in A_3 \ \& \ \ldots\}$$

$$= \{f : \mathbb{Z}_+ \to \bigcup_{i \in \mathbb{Z}_+} A_i \mid \text{for all } i \in \mathbb{Z}_+ \ a_i \in A_i\}$$

Give an example of an element of $\underbrace{\prod_{n \in \mathbb{Z}_+} A_n \text{ where } A_n =}_{\text{for all } n \in \mathbb{Z}_+}$ $\underbrace{(1/n^3, 1/n) \subset \mathbb{R}}_{\text{for all } n \in \mathbb{Z}_+}$.

That is give an example of $(a_1, a_2, a_3, \ldots)$ where $a_n \in \left(\frac{1}{n^3}, \frac{1}{n}\right)$. $a_n = \frac{1}{n^2}$ works, except at $n=1$

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$A_1 = (1,1) = \{x \in \mathbb{R} \mid 1 < x < 1\} = \emptyset$

The first coordinate can't be in A_1 because nothing is in A_1.

Therefore, $\emptyset = \prod_{n \in \mathbb{Z}_+} A_n$.

But $\left(\frac{1}{n^2}\right)_{n=2}^{\infty} \in \prod_{n=2}^{\infty} A_n = A_2 \times A_3 \times \ldots\ldots$

We will assume that if $A_1, A_2, A_3, \ldots\ldots \neq \emptyset$, then there exists $(a_1, a_2, a_3, \ldots\ldots) \in \prod_{n=1}^{\infty} A_n$.

(This is equivalent to something called the axiom of choice).

~~choice~~ countable choice

If $\mathcal{A}$ is a set of sets and $\mathcal{A} \neq \emptyset$ then $\prod_{A \in \mathcal{A}} A = \{f : \mathcal{A} \to \bigcup \mathcal{A} \mid f(A) \in A \text{ for all } A \in \mathcal{A}\}$

This is equivalent to the (full) Axiom of Choice: If $A \neq \emptyset$ for all $A \in \mathcal{A}$, and and $\mathcal{A} \neq \emptyset$, then $\prod_{A \in \mathcal{A}} A \neq \emptyset$.

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Suppose $f: \mathcal{A} \to \cancel{\mathcal{B}}$ and I write $B_A = f(A)$ for all $A \in \mathcal{A}$. Then....

$$\prod_{A \in \mathcal{A}} B_A = \{ g: \mathcal{A} \to \bigcup \mathcal{B} \mid g(A) \in B_A \text{ for all } A \in \mathcal{A} \}$$

$$\bigcup_{A \in \mathcal{A}} B_A = \{ b \mid \text{for some } A \in \mathcal{A}, b \in B_A \}.$$

$$\bigcap_{A \in \mathcal{A}} B_A = \{ b \mid \text{for all } A \in \mathcal{A}, b \in B_A \}$$

for all $r \in \mathbb{R}$, $B_r = (r-1, r+1) \subset \mathbb{R}$.

$\bigcup_{r \in \mathbb{R}} B_r = \mathbb{R}$ because $r \in B_r$ for all $r \in \mathbb{R}$, so every $r \in \mathbb{R}$ is in B_s for some $s \in \mathbb{R}$, namely $s = r$. (And, conversely, if $t \in \bigcup_{r \in \mathbb{R}} B_r$ then $t \in B_x$ for some $x \in \mathbb{R}$, so $t \in \mathbb{R}$.)

$$\bigcap_{x \in \mathbb{R}} (-1-x^2, x^2+1) \subset (-1-0^2, 0^2+1)$$ by definition of $\bigcap$

$= (-1,1)$. Also, $(-1,1) \subset \bigcap_{x \in \mathbb{R}} (-1-x^2, x^2+1)$ because $y \in (-1,1) \Leftrightarrow -1 < y < 1$

$\Rightarrow -1-x^2 \leq -1 < y < 1 \leq 1+x^2$ for all $x \in \mathbb{R}$

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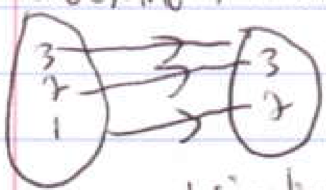
$\Rightarrow -1-x^2 < y < 1+x^2$ for all $x \in \mathbb{R}$.

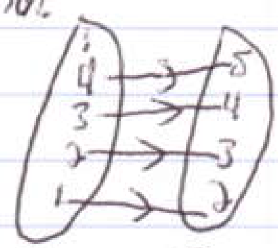
So, $(-1,1) = \bigcap_{x \in \mathbb{R}} (-1-x^2, 1+x^2)$.

Dedekind-infinite: There is a bijection between A & $A - \{a\}$ for some $a \in A$.

Dedekind-finite: There is no such bijection.



no bijection



$\mathbb{Z}_+$ $\mathbb{Z}_+ - \{1\}$

A set F is finite if there is a bijection $\{1, \dots, n\} \to F$ for some $n \in \mathbb{Z}_+$ (or $F = \emptyset$).

A set is infinite if it's not finite.

Corollary 6.3: finite $\Rightarrow$ Dedekind finite.

Hard Theorem: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Easy Corollary $\sum_{n=1}^{\infty} \frac{6}{n^2} = \pi^2$

Every finite nonempty set has a bijection from a unique ~~$n \in \mathbb{Z}$~~ $\{1, \dots, n\}$. (where $n \in \mathbb{Z}_+$)

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This is because if $m \neq n$, then there is no bijection $\{1, \ldots, n\} \to \{1, \ldots, m\}$