9/23/10

Reading Assignment for Tuesday
pages 75-86 (§ 12-14)

There will an easy quiz to
check if you ~~have~~ read these.

$f(1) = 1$

$f(n+1) = \sqrt{f(1) + f(2) + \dots + f(n)}$

This define a function $f: \mathbb{Z}_+ \to \mathbb{R}$

n	$f(n)$
1	1
2	$\sqrt{1} = 1$
3	$\sqrt{1+1} = \sqrt{2}$
4	$\sqrt{1+1+\sqrt{2}} = \sqrt{2+\sqrt{2}}$
5	$\sqrt{1+1+\sqrt{2}+\sqrt{2+\sqrt{2}}} = \sqrt{2+\sqrt{2}+\sqrt{2+\sqrt{2}}}$
6	

$f(5) = f(4+1) = \sqrt{f(1)+f(2)+f(3)+f(4)}$

$= \sqrt{1 + \sqrt{f(1)} + \sqrt{f(1)+f(2)} + \sqrt{f(1)+f(2)+f(3)}}$

$= \sqrt{1+\sqrt{1} + \sqrt{1+\sqrt{f(1)}} + \sqrt{1+\sqrt{f(1)}+\sqrt{f(1)+f(2)}}}$

$= \sqrt{1+\sqrt{1} + \sqrt{1+\sqrt{1}} + \sqrt{1+\sqrt{1}+\sqrt{1+\sqrt{1}}}}$

Notation

A^{ω} = set of functions $f: \mathbb{Z}_+ \to A$
= set of infinte sequences in A.

$A^{<\omega}$ = [set of finite sequences in A]

$= \{\emptyset\} \cup A^2 \cup A^3 \cup A^4 \cup \ldots =$

$$\underbrace{\{\emptyset\}}_{A^0} \cup \bigcup_{n \in \mathbb{Z}_+} A^n = \bigcup_{n \in \{0\} \cup \mathbb{Z}_+} A^n$$

Principle of recursive Defimition

if $G: A^{<\omega} \to A$, then there is a unique function $f \in A^{\omega}$ such that

$f(n) = G(f(1), f(2), \ldots f(n-1))$

for a $n \in \mathbb{Z}_+$.

$G: [0, \infty)^{<\omega} \to [0, \infty]$

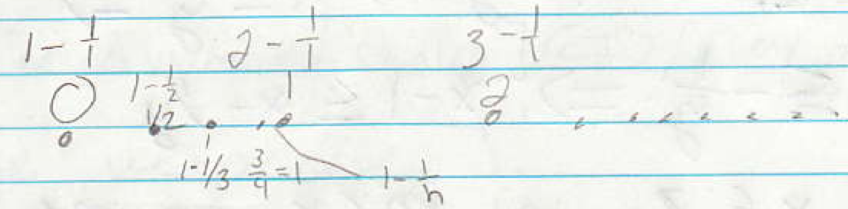
$G(\emptyset) = 1$

$G((x_1, \ldots x_n)) = \sqrt{x_1 + x_2 + \ldots + x_n}$ (if $n > 0$)

There is a unique $f: \mathbb{Z}_+ \to [0, \infty)$
(ie, $f \in [0, \infty)^{\omega}$) such that
$f(n) = G((f(1), \ldots, f(n-1)))$ forall $n \in \mathbb{Z}_+$

- A order relation $<$ on a set L is a well-ordering if every nonempty subset of L has a minimum.
- An order relation is a well ordering $\iff$ there is no infinite desending sequence: $x_1 > x_2 > x_3 > x_4 > x_5 > \dots > x_n >$

$A = \{m - \frac{1}{n} \mid m, n \in \mathbb{Z}_+\} \subset \mathbb{R}$



<u>A =is well ordred</u>

Let $\emptyset \neq B \subset A$.

$C = \{m \in \mathbb{Z}_+ / m - \frac{1}{n} \in B$ for some $n \in \mathbb{Z}_+\}$

is not $\emptyset$ because $B \neq \emptyset$.

Let $m = \min(C)$

$\min(C)$ exists because $C \subset \mathbb{Z}_+$

$D = \{n \in \mathbb{Z}_+ \mid m - \frac{1}{n} \in B\}$.

$n = \min(D)$ (ok because $\emptyset \neq D \subset \mathbb{Z}_+$)

$m \in C \Rightarrow m - \frac{1}{k} \in B$ for some $k \Rightarrow k \in D$ for some k

Claim $m - \frac{1}{n} = \min(B)$.

Suppose $x, y \in \mathbb{Z}$ and $x - 1/y \in B$.

$x \in C$. $x \geq \min(C) = m$

Case 1 $x > m$:

$y \in \mathbb{Z}_+ \Rightarrow 1 \leq y \Rightarrow 1 \geq \frac{1}{y} \Rightarrow$

$-1 \leq -\frac{1}{y} \Rightarrow x - 1 \leq x - 1/y$

$m, x \in \mathbb{Z}_+$ and $m < x$, so $m \leq x-1$.

So, $m \leq x-1 \leq x - \frac{1}{y}$. So,

$m - \frac{1}{n} \leq m \leq x - \frac{1}{y}$.

Case 2 $x = m$:

$m - \frac{1}{y} = x - \frac{1}{y} \in B$ and $y \in \mathbb{Z}_+$, so

$y \in D$. So, $y \geq \min(D) = n$.

$y \geq n > 0 \Rightarrow \frac{1}{y} \leq \frac{1}{n} \Rightarrow -\frac{1}{y} \geq -\frac{1}{n}$

$\Rightarrow x - \frac{1}{y} = m - \frac{1}{y} \geq m - \frac{1}{n}$ ✓

Same argument proves

$\mathbb{Z}_+ \times \mathbb{Z}_+$, $<$ dictionary is we-ordered.

Is the an order relation $\sqsubset$ on $\mathbb{R}$

that is a well-ordering? $\sqsubset$ can't be $<$!

(the usual ordering)!

$\mathbb{R} = \{x_1 \sqsubset x_2 \sqsubset x_3 \sqsubset x_4 \sqsubset x_5 \sqsubset \dots \sqsubset x_n \sqsubset \dots \sqsubset x_\omega \sqsubset x_{\omega+1} \dots\}$??

Yes, but it "infinitely complicated"!

[The Axiom of choice] $\Longleftrightarrow$ [Every set can

be we-ordered].

Another counterintutive object

S_Ω is an uncountable well-ordered set but

for all $x \in S_\Omega$, $\{y \in S_\Omega \mid y < x\}$ is

countable.

$S_\Omega = \{1 < 2 < 3 < 4 < \dots n < n+1 < \dots < \omega < \omega+1 < \omega+2$

$< \dots \omega+\omega < \omega+\omega+1 < \omega+\omega+2$

$< \omega+\omega+\omega = \omega \cdot 3 < \dots < \omega \cdot 4 < \dots < \omega \cdot 5$

$< \dots < \omega \cdot \omega = \omega^2 < \dots < \omega^2+\omega < \dots < \omega^2 \cdot 2$

$< \dots < \omega^3 < \dots \omega^4 < \dots < \omega^{\omega^\omega} < \dots \omega^{\omega^{\omega^\omega}}$

$< \dots \varepsilon_0 < \varepsilon_0+1 < \dots\}$

You can extend the principle of recursive definition from $\mathbb{Z}_+$ to any well-ordered set W.

Before $f(1) = a$

$$f(n+1) = G((f(1), \dots f(n))$$

Now: $f(\min(W)) = a$

$$f(x) = G((f(y) : y < x))$$