9/28/10

What is the definition of topology?

A set $\mathcal{T} \subset P(X)$ for some set X

where $\emptyset, X \in \mathcal{T}$, $\forall$ arb $\subset \mathcal{T}$

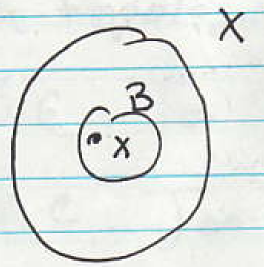
$\bigcup$ arb $\in \mathcal{T}$, and $\forall$ finite $\mathcal{F} \in \mathcal{T}$ $\cap \mathcal{F} \in \mathcal{T}$.

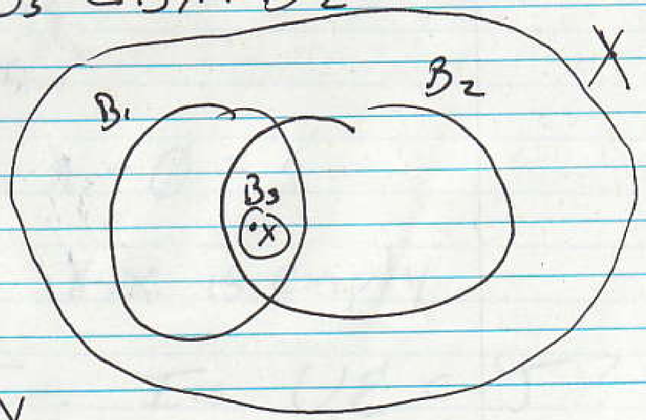
- $\mathcal{B}$ is a basis for X if • $\forall x \in X\ \exists B \in \mathcal{B}$

  $x \in B$

- $\forall B_1, B_2 \in \mathcal{B}$ $\forall x \in B_1 \cap B_2$

  $\exists B_3 \in B$ $x \in B_3 \subset B_1 \cap B_2$





- $\forall B \in \mathcal{B}$ $B \subset X$

$X = \{1, 2, 3, 4\}$

$\mathcal{B} = \{\{1\}, \{2,3\}, \{3\}, \{3,4\}\}$

is a basis of X.

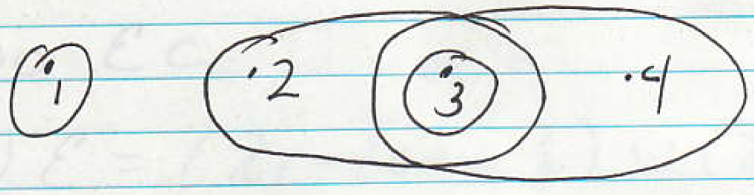
$\{2,3\}, \{3,4\} \in \mathcal{B}$

$\mathcal{B} \in \{2,3\} \cap \{3,4\}$

$3 \in \{3\} \subset \{2,3\} \cap \{3,4\}$

$\{3\} \in \mathcal{B}$

$\{1\} \cap \{3,4\} = \emptyset \in$ (is ok)



Let X be an infinite set. Is the collection
$\mathcal{T}_\infty = \{U \subset X \mid X - U$ is infinite or empty or $= X\}$
a topology on X?

$\emptyset \in \mathcal{T}_\infty$? Yes $X - \emptyset = X$

$X \in \mathcal{T}_\infty$? Yes $X - X$ is empty.

Let $\mathcal{C} \subset \mathcal{T}_\infty$. Is $\bigcup \mathcal{C} \in \mathcal{T}$?

Is $X - \bigcup \mathcal{C}$ infinite, $\emptyset$, or X?

$$X - \bigcup \mathcal{C} = X - \bigcup_{U \in \mathcal{C}} U = \bigcap_{U \in \mathcal{C}} (X - U)$$

Each of $X - U$ is infinite, $\emptyset$ or X

because each $U \in \mathcal{E} \subset \mathcal{T}$.

Suppose $X = \mathbb{Z}_+$. $\mathcal{E} = \{\mathbb{Z}_+ - \{\text{primes}\},$
$\mathbb{Z}_+ - \{\text{evens}\}\}$.

Since {evens} and {primes} are ∞
subsets of $\mathbb{Z}_+$, $\mathbb{Z}_+ - \{\text{evens}\}$, $\mathbb{Z}_+ - \{\text{primes}\} \in \mathcal{T}_\infty$
so $\mathcal{E} \subset \mathcal{T}$.

$\bigcup \mathcal{E} = (\mathbb{Z}_+ - \{\text{evens}\}) \cup (\mathbb{Z}_+ - \{\text{primes}\})$
$= \mathbb{Z}_+ - (\{\text{evens}\} \cap \{\text{primes}\})$

$= \mathbb{Z}_+ - \{2\} \Rightarrow \mathbb{Z}_+ - \bigcup \mathcal{E} = \{2\} \Rightarrow \bigcup \mathcal{E} \notin \mathcal{T}_\infty$

- Prove that the standard topology on $\mathbb{R}$ is finer than the finite complement topology.

Eg. $(1,5)$ is open;
$(2,3) \cup (-7,-6)$ is open
$\bigcup_{n \in \mathbb{Z}_+} (n - 2^{-n}, n^2)$ is open
in the standard topology of $\mathbb{R}$.

*

Also, $\bigcup_{n \in \mathbb{Z}_+} (3n+1, \; 3n+2)$

Proof: Assume $U \in \mathcal{T}_f$

Prove $U \in \mathcal{T}$ where $\mathcal{T}$ is the standard topology on $\mathbb{R}$. Case

$\mathbb{R} - U = \mathbb{R}$

$$U = \mathbb{R} - (\mathbb{R} - U) = \mathbb{R} - \mathbb{R} = \phi \in \mathcal{T},$$

Express $\mathbb{R} - \{3\}$ as a union of open intervals. $(-\infty, 3) \cup (3, \infty)$.

Case 3 $\mathbb{R} - U$ is finite.

Let $\mathbb{R} - U = \{a_1 < a_2 < a_3 < \dots < a_n\}$

$\mathbb{R} - U = (-\infty, a_1) \cup (a_1, a_2) \cup \dots (a_{n-1}, a_n) \cup$

$(a_n, \infty) \in \mathcal{T}$.

We proved $\mathcal{T}_f \subset \mathcal{T}$.

Lets prove $\mathcal{T}_f \neq \mathcal{T}$.

How? Find ~~a~~ $U \in \mathcal{T} - \mathcal{T}_f$.

$\mathbb{R} - (\mathbb{R} - \mathbb{Z}) = \mathbb{Z}$ not finite or $\mathbb{R}$.

$\mathbb{R} - \mathbb{Z} \notin \mathcal{T}_f$ ✓

$(2,3) \in \mathcal{T}$

$\mathbb{R} - (2,3) = (-\infty\ 2] \cup [3, \infty)$ is not finite or $\mathbb{R}$. $(2,3) \notin \mathcal{T}_f$

Read by Thurs.

§ 14, 15, 16 p. 84 - 91