Thm. 1. $\{(x,y)\in\mathbb{R}^2 \mid x<y\}$ is open (in the product topology, which is the standard topology on $\mathbb{R}^2$).

Pf. Suppose $(x,y)\in U=\{(a,b)\in\mathbb{R}^2 \mid a<b\}$. By Example 1 of §15,
$\mathcal{B}=\{(a,b)\times(c,d) \mid a<b \ \& \ c<d \ \& \ a,b,c,d\in\mathbb{R}\}$
is a basis for $\mathbb{R}^2$ (with the standard topology). Therefore, U is open if and only if for all $(x,y)\in U$, there exists $(a,b)\times(c,d)\in\mathcal{B}$ such that $(x,y)\in(a,b)\times(c,d)\subset U$.
Choose reals a, b, c such that (see Figure
1.) $a<x<b<y<c$. (For example, $a=x-1$, $b=\frac{x+y}{2}$, $c=y+1$.) Then $x\in(a,b)$ & $y\in(b,c)$, so $(x,y)\in(a,b)\times(b,c)\in\mathcal{B}$. Also, $(a,b)\times(b,c)\subset U$ because if $(z,w)\in(a,b)\times(b,c)$, then $a<z<b$ & $b<w<c$, so $z<b<w$, so $z<w$, so $(z,w)\in U$. □

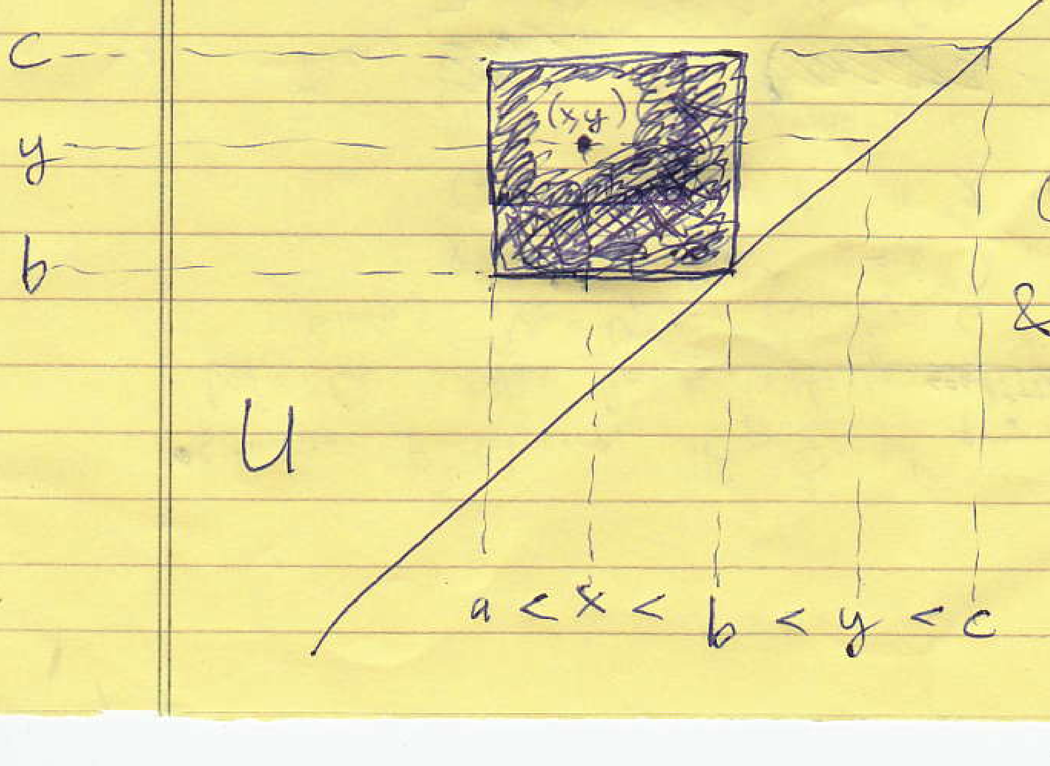

Figure 1: $(x,y)\in(a,b)\times(b,c)$ & $(a,b)\times(b,c)\subset U$.

Thm. 2.

If $Y = [0,1) \cup \{2\} \subset \mathbb{R}$, then $\{2\} \in \mathcal{T}_S$ = the subspace topology on Y from $\mathbb{R}$ and $\{2\} \notin \mathcal{T}_O$ = ther order topology on Y.

Pf. $\{2\} = (1,3) \cap Y$ and $(1,3)$ is open in $\mathbb{R}$. Hence, $\{2\} \in \mathcal{T}_S$.

~~By definition of order topology, $\{2\}$ is in $\mathcal{T}_O$ is open if and only if it is in the top~~ $\mathcal{T}_O$ is generated by the basis $\mathcal{B}$ consisting of sets of the form $\{y \in Y \mid 0 \leq y < b\}$, $\{y \in Y \mid a < y \leq 2\}$, or $\{y \in Y \mid a < y < b\}$ where $a, b \in Y$ and $a < b$. Seeking a contradiction, suppose that $\{2\} \in \mathcal{T}_O$. Then, since $2 \in \{2\}$, there is some $B \in \mathcal{B}$ such that $2 \in B \subset \{2\}$. So, $B = \{2\}$. Therefore, B is not of the form $\{y \in Y \mid 0 \leq y < b\}$ (because $0 \notin B$). Also, B is not of the form $\{y \in Y \mid a < y < b\}$ because if $b \in Y$, then $b \leq 2$, so $2 \in B$ would imply $a < 2 < b \leq 2$ if $B = \{y \in Y \mid a < y < b\}$, and $2 < b \leq 2$ is impossible.

Finally, B is not of the form $\{y \in Y \mid a < y \leq 2\}$ because if (see Figure 2) $a, b \in Y$ & $a < b$, then $a < b \leq 2$, so $a < 2$, which implies $a < 1$ (since $[1, 2) \cap Y = \emptyset$), which implies $(a+1)/2 \in \{y \in Y \mid a < y \leq 2\}$ (because $a < (a+1)/2 < 1$), which would imply the impossible $(a+1)/2 \in \{2\}$ if $\{2\} = B = \{y \in Y \mid a < y \leq 2\}$.

This is our desired contradiction, for we have shown that B, supposedly in $\mathcal{B}$, is not in any of the three forms taken by sets in $\mathcal{B}$. $\square$

Figure 2.

$$[0 \qquad \underbrace{(\overset{\bullet}{a} \quad \overset{\bullet}{\tfrac{a+1}{2}} \qquad 1) \quad \cup \quad \{2\}\;2]}$$

$$\frac{a+1}{2} \in \{y \in Y \mid a < y \leq 2\}$$