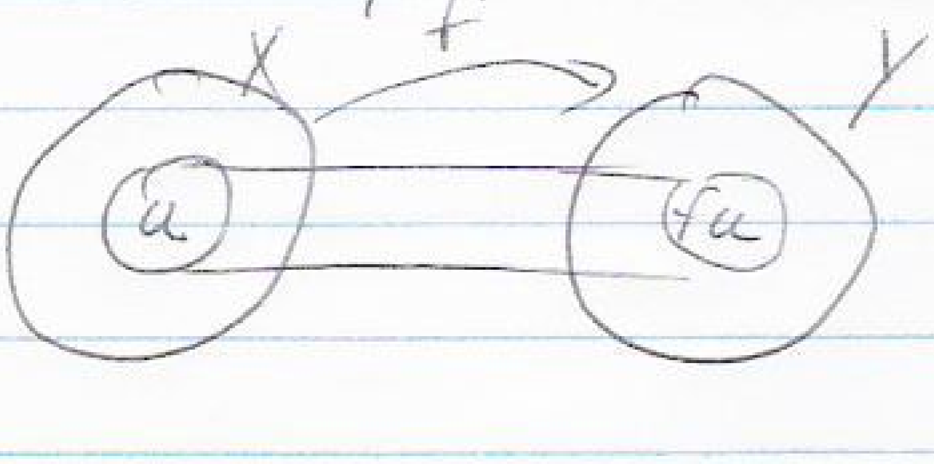
Notes

10-12-10

Homework

7	4
10	1
13	1
13	8a
16	4
17	13

§16 #4



f is open if u open $\Rightarrow f(u)$ open

$= \{f(x) \mid x \in u\}$

$= \{y \in Y \mid \exists x \in u \ f(x) = y\}$

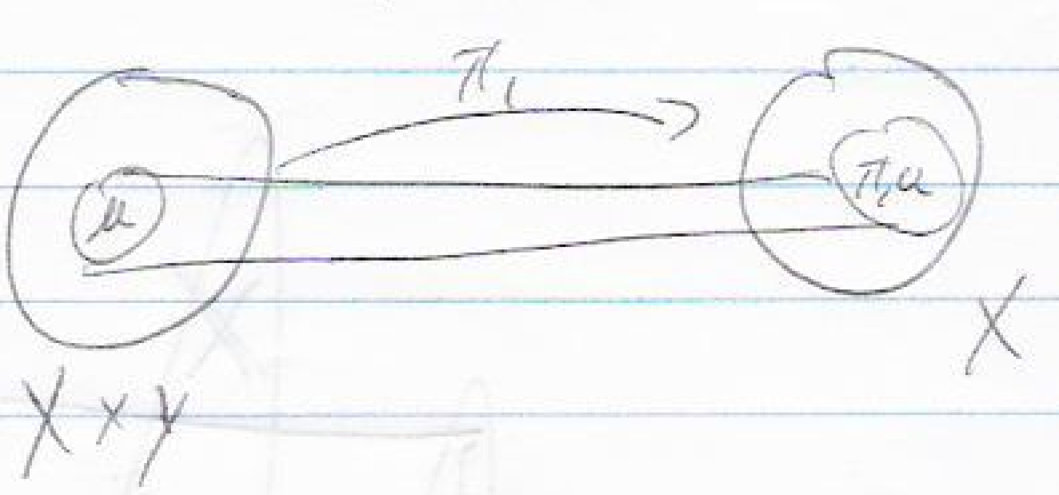
there exists

Projections §15

$\pi_1 : X \times Y \to X \qquad \pi_1(x,y) = x$

$\pi_2 : X \times Y \to Y \qquad \pi_2(x,y) = y$

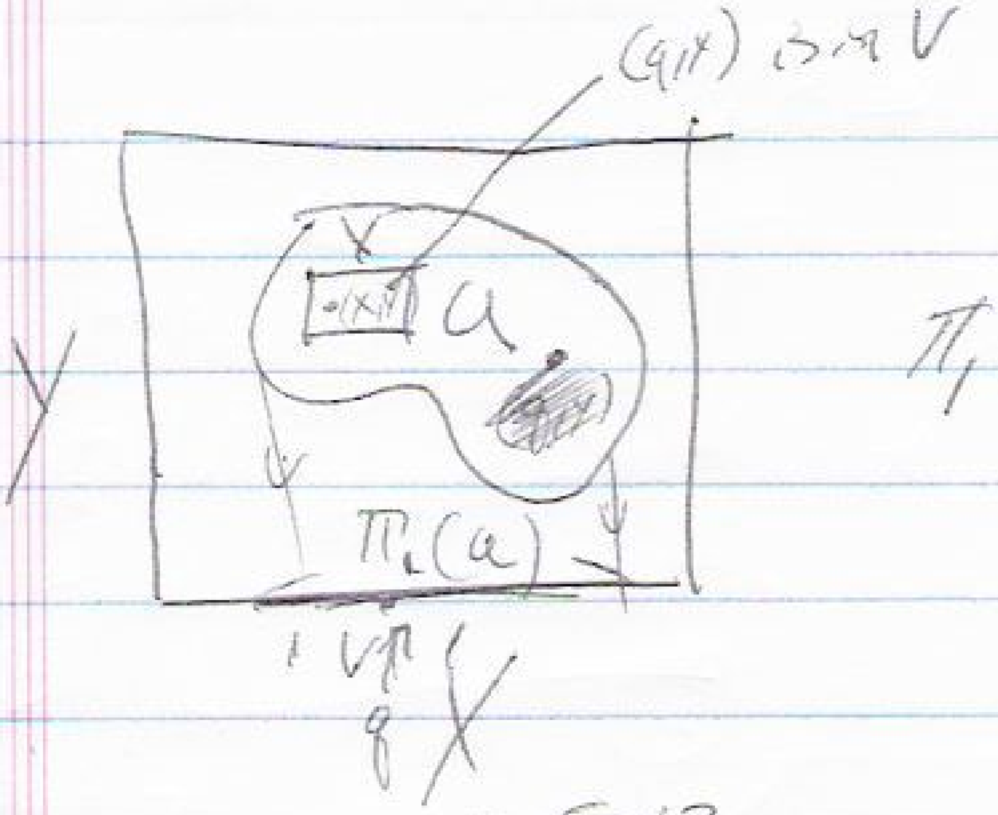
Show π_1 & π_2 are open.



u open $\Rightarrow \pi_1(u)$ open

$B = \{V \times W \mid V \text{ open}, W \text{ open}\}$ is a basis for $X \times Y$

Assume $u \subset X \times Y$ is open.
We need to prove $\pi_1(u)$ is open



$$\pi_1(U) = \{X \mid (X,Y) \in U\}$$

§13

☆ By definition, for all $(X,Y) \in U$, there exists $V \times W \in B$ such that

$$(X,Y) \in V \times W \subset U.$$

For every $X \in \pi_1(U)$, pick $Y_X \in Y$ such that $(X, Y_X) \in U$. Since B is a basis for $X \times Y$ & U is open in $X \times Y$, $(X, Y_X) \in V_X \times W_X \subset U$ for some open V_X & W_X we can choose.

Then $\bigcup_{X \in \pi_1(U)} V_X$ is open. So, it's enough to prove.

$\pi_1(U) = \bigcup_{X \in \pi_1(U)} V_X$ So show

① $p \in \pi_1(U) \Rightarrow p \in \bigcup_{X \in \pi_1(U)} V_X$.

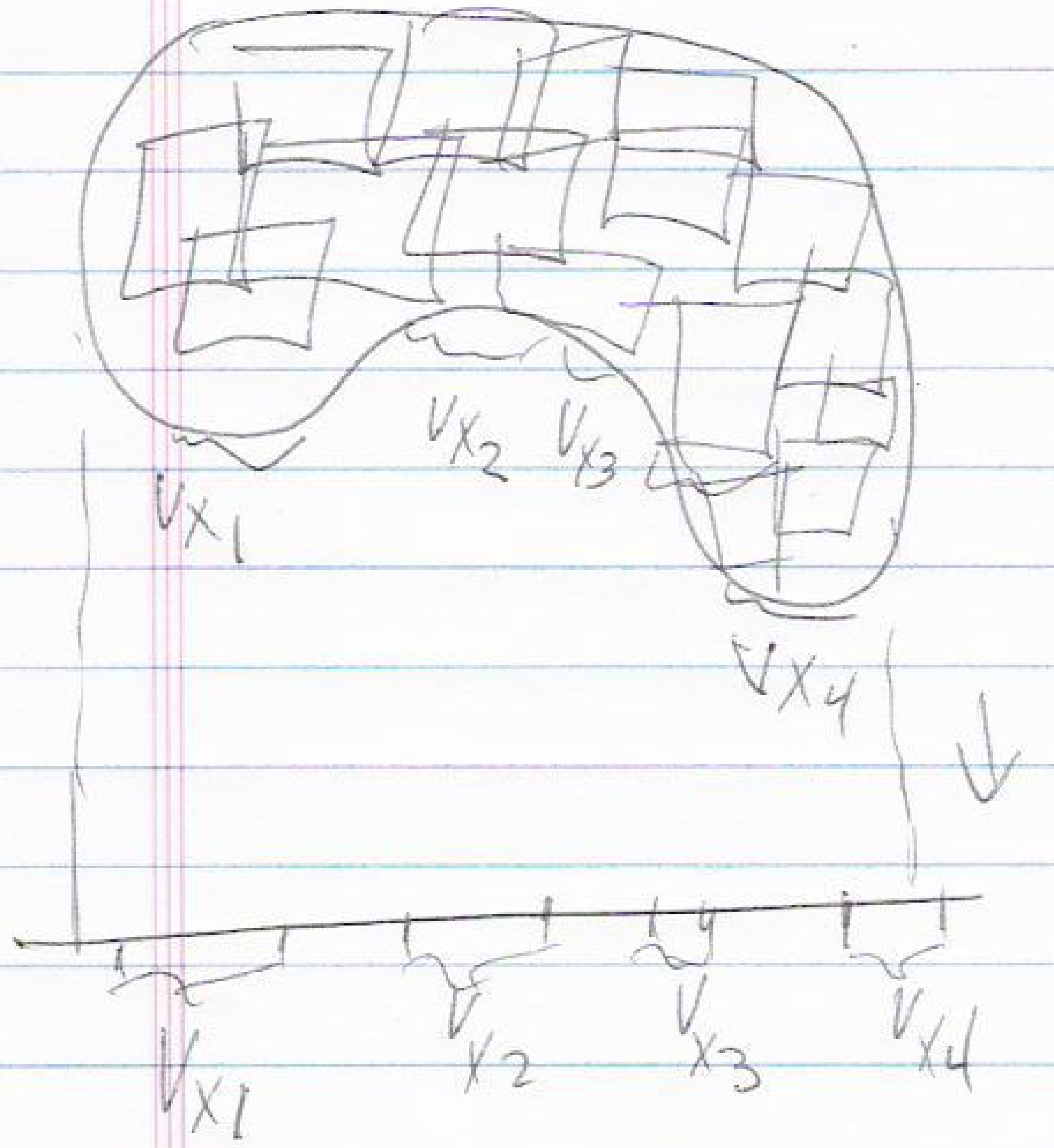
② $q \in \bigcup_{X \in \pi_1(U)} V_X \Rightarrow q \in \pi_1(U)$.

1 $p \in \pi_1(\mathcal{U}) \Rightarrow (p, y_p) \in U_p \times W_p \Rightarrow p \in U_p$

$\Rightarrow p \in \bigcup_{x \in \pi_1(\mathcal{U})} U_x$

2 $q \in \bigcup_{x \in \pi_1(\mathcal{U})} U_x \Leftrightarrow \exists x_{\in \pi_1(\mathcal{U})}\ q \in U_x$

~~Pick~~ pick such an x. Then $(q, y_x) \in U_x \times W_x \subset \mathcal{U}$
so $(q, y_x) \in \mathcal{U}$, so $q \in \pi_1(\mathcal{U})$. $\square$ ✓



Lemma 13.1 A set is open exactly when it is a union of basic sets.

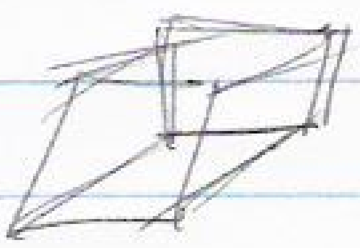
$\mathcal{U} = \bigcup_{j \in J} (U_j \times W_j)$ where U_j, W_j open

$\pi_1(\mathcal{U}) = \bigcup_{j \in J} \pi_1(U_j \times W_j) = \bigcup_{j \in J} U_j$ is open.

§ 19 The Product Topology

Product spaces with many factors.

$\{U \times V \times W \mid U, V, W \text{ open}\}$ is a basis of $X \times Y \times Z$



$\{U_1 \times U_2 \times \cdots \times U_n \mid U_1, \ldots, U_n \text{ open}\}$ is a basis of $X_1 \times X_2 \times \cdots \times X_n$

$\{\prod_{\alpha \in J} U_\alpha \mid \forall \alpha \in J\ U_\alpha$ open and for only finitely many α's is $X_\alpha \neq U_\alpha\}$

is a basis for $\prod_{\alpha \in J} X_\alpha$

$\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} \mathbb{R} = \{f \mid f : \mathbb{Z}_+ \to \mathbb{R}\}$

$\mathbb{R} \times (0,1) \times (2,3) \times \mathbb{R} \times \mathbb{R} \times (-7,11) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cdots$ All $\mathbb{R}$s

$= \{f \in \mathbb{R}^\omega \mid f(2) \in (0,1), f(3) \in (2,3) \text{ and } f(5) = (-7,11)\}$

$\square_{\alpha \in J} X_\alpha$ stands for $\prod_{\alpha \in J} X_\alpha$ with the box product topology, where $\{\prod_{\alpha \in J} U_\alpha \mid \forall_\alpha \in J\ U_\alpha \text{ open}\}$

is a basis. In $\square_{n \in \mathbb{Z}_+} \mathbb{R}$, $(0,1) \times (0,1/2) \times (0,1/3) \times (0,1/4)$

is open

$f_1, f_2, f_3, f_4 \ldots \in \mathbb{R}^\omega$

$f_1 = (1, 0, 0, 0, \ldots)$

$f_2 = (0\ 1/2, 0, 0, 0 \ldots)$

$f_n = (\underbrace{0, 0 \ldots 0}_{n-1}\ 1/n, 0, 0, 0)$

$g = (0, 0, 0, 0, \ldots)$

Does $(f_n)_{n \in \mathbb{Z}_+}$ converse to g?

Not in the box product topology:

$g \in U = (-1, 1) \times (-1, 1/2) \times (-1, 1/3) \times (-1, 1/4) \times \cdots$

U is open, but $f_n \notin$ for all $n \in \mathbb{Z}_+$

why? $f_n(n) = 1/n \notin (-1, 1/n)$

In the product topology,

If V is a nbhd of g,

$0 \in (a_1, b_1) \ldots$
$0 \in (a_n, b_n)$

Then $g \in (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ldots$
$\subset V$

So except possibly $f_1, \ldots f_n$, every f_K is in V:

$n < K \Rightarrow f_K = (\underbrace{0, \ldots 0}_{K-1 \geq n}, 1/K, 0, 0, 0 \ldots)$

$\in \mathbb{R}$ $\quad \in \mathbb{R}$

$0 \in n(a_1, b_1)$

So, $(f_n)_{n \in \mathbb{Z}_+}$ converges to g.