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Notes for 10-14-10 (Review)

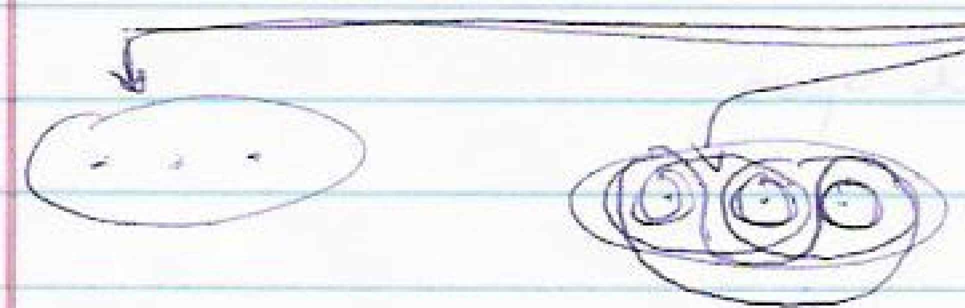
Section 12

A topology on X is a set $\mathcal{T} \subset P(X)$ with $\phi, X \in \mathcal{T}$ such that $U_1, \ldots, U_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$ and $\mathcal{A} \subset \mathcal{T} \Rightarrow \bigcup \mathcal{A} \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets. Often we write just X if it's understood what $\mathcal{T}$ should be.

$\{\phi, X\}$ = indiscrete topology on X.

$P(X)$ = discrete topology on X.



$\mathcal{T} \subset \mathcal{T}' \iff \mathcal{T}$ coarser than $\mathcal{T}'$ $\iff \mathcal{T}'$ finer than $\mathcal{T}$. $\Big\}$ $\mathcal{T}, \mathcal{T}'$ top's on some set.

Is $\{\phi, \{1\}, \{1,2\}, \{1,2,3\}, \{2,3\}\}$ a topology on $\{1,2,3\}$? Why? No, since $\{1,2\} \cap \{2,3\} = \{2\} \notin$

Section 17

C closed $\iff X - C$ is open $\iff C = \overline{C}$ ~~$\Leftarrow [\forall p \in C$~~

~~$\forall U$ nbhd~~ . $p \in \overline{A} \iff \forall U$ nbhd of p, $A \cap U \neq \phi$.

$\iff p \in \bigcap \{B \mid A \subset B \text{ and } B \text{ is closed}\}$.

Two equivalent definition of closure

$p \in \operatorname{Int} A \iff \exists\, U$ nhbd of p $U \subset A$.

$\iff p \in \bigcup\{B \mid B \subset A$ and B is open$\}$.

A is open $\iff A = \operatorname{Int} A$.

$C_1, C_2, \ldots C_n$ closed $\Rightarrow \bigcup_{i=1}^{n} C_i$ closed

$\forall C \in \mathcal{C}$ C closed $\Rightarrow \bigcap \mathcal{C}$ closed.

$\emptyset$, X are closed (where X is the ~~univer~~ whole space)

$U_1, \ldots, U_n$ open $\Rightarrow \bigcap_{i=1}^{n}$ open.

$\forall U \in \mathcal{U}$ U open $\Rightarrow \bigcup \mathcal{U}$ open.

$\emptyset$, X are open.

U nhbd of p means $p \in U$ and U is open.

$p \in A' = \{$limit points of $A\} \iff \forall (U$ nhbd of $p)$ $\emptyset \neq A \cap U \neq \{p\}$.

E.g. 0 is a limit point of $\{1/n \mid n \in \mathbb{Z}_+\}$ and 0 is a limit point of $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$. However, 0 is not a limit pt. of $\{0\} \cup [1,2]$, nor is 0 a limit pt. of $[1,2]$. Think "infinitely close", ~~but not just~~ to $A - \{p\}$. ~~to itself~~ $\bar{A} = A \cup A'$. A is closed $\iff A' \subset A \iff \bar{A} = A$.

What is the closure of $(2,3) \cup (3,4) \cup [5,6)$?

$[2,4] \cup [5,6]$.

What are the limit pts of $\{1\} \cup (2,3)$? $[2,3]$.

Think 1 is not ∞ close to $(\{1\} \cup (2,3)) - \{1\}$.

More precise: $(1/2, 3/2)$ misses $(2,3)$.

Hausdorff: distinct points have disjoint neighborhoods $\odot$ $\odot$.

T_1: finite sets are closed. Hausdorff $\Rightarrow T_1$.

Section 10

An ordered set A is well ordered if $\emptyset \neq B \subset A \Rightarrow \exists \min(B)$.

$\mathbb{Z}_+$ is well ordered, so is $\mathbb{Z}_+^2$, $<$ dict.

S_Ω is an uncountable well-ordered set.

~~A subset T of S_Ω~~ Every $T \subset S_\Omega$ that has an upper bound is ctbl. Conversely, Every ctbl $T \subset S_\Omega$ has an upper bound.

Section 7

A countably infinite $\iff \exists f: \mathbb{Z}_+ \to A$ bijective.
$\iff \exists g: A \to \mathbb{Z}_+$ bijective.

A countable $\iff$ (A is finite or countably infinite)
$\iff \exists f: \mathbb{Z}_+ \to A$ surjective
$\iff \exists g: A \to \mathbb{Z}_+$ injective

A finite $\iff \exists n \in \mathbb{Z}_+ \; \exists h: \{1, \ldots, n-1\} \to A$ bijective

Countability is preserved by finite products, countable unions, subsets, images of ~~surjections~~ functions, but not always preserved by infinite products.

If A is ctbl and $f: A \to B$, then $f(A)$ is ctbl.

$\prod_{n \in \mathbb{Z}_+} \{0,1\} = \{0,1\}^\omega$ is uncountable

Prove the $\mathbb{Z}$ is countable.

There is a bijective $f: \mathbb{Z} \to \mathbb{Z}_+$

negatives $\to$ evens
0 $\to$ } odds
positive $\to$

$$f(x) = \begin{cases} -2x & x < 0 \\ 2x+1 & x \geq 0 \end{cases}$$

If $f(x) = f(y)$, then $f(x)$ and $f(y)$ are both even or both odd.

both even: $-2x = -2y \Rightarrow x = y$

both odd: $2x+1 = 2y+1 \Rightarrow x = y$.

So, f is injective.

If $n \in \mathbb{Z}_+$ is even, then $n = 2m$ for some $m \in \mathbb{Z}_+$,

so $n = -2(-m) = f(\underbrace{-m}_{\in \mathbb{Z}})$. If $n \in \mathbb{Z}_+$ is odd, then

$n = 2m+1$ for some $m \in \mathbb{Z}_+ \cup \{0\}$, so $n = f(m)$.

Milovich adds:

An easier proof of $\mathbb{Z}$'s countability:

$\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{-n \mid n \in \mathbb{Z}_+\}$.
Since a finite union of countable sets is countable and $id_{\mathbb{Z}_+} : \mathbb{Z}_+ \to \mathbb{Z}_+$ and $f : \mathbb{Z}_+ \to \{0\}$, $n \mapsto 0$ are clearly surjections, witnessing that $\mathbb{Z}_+$ & $\{0\}$ are countable, it suffices to prove that $X = \{-n \mid n \in \mathbb{Z}_+\}$ is countable, which is done by noting that $g : \mathbb{Z}_+ \to X$, $n \mapsto -n$ is clearly surjective.