Oct 28, 2010

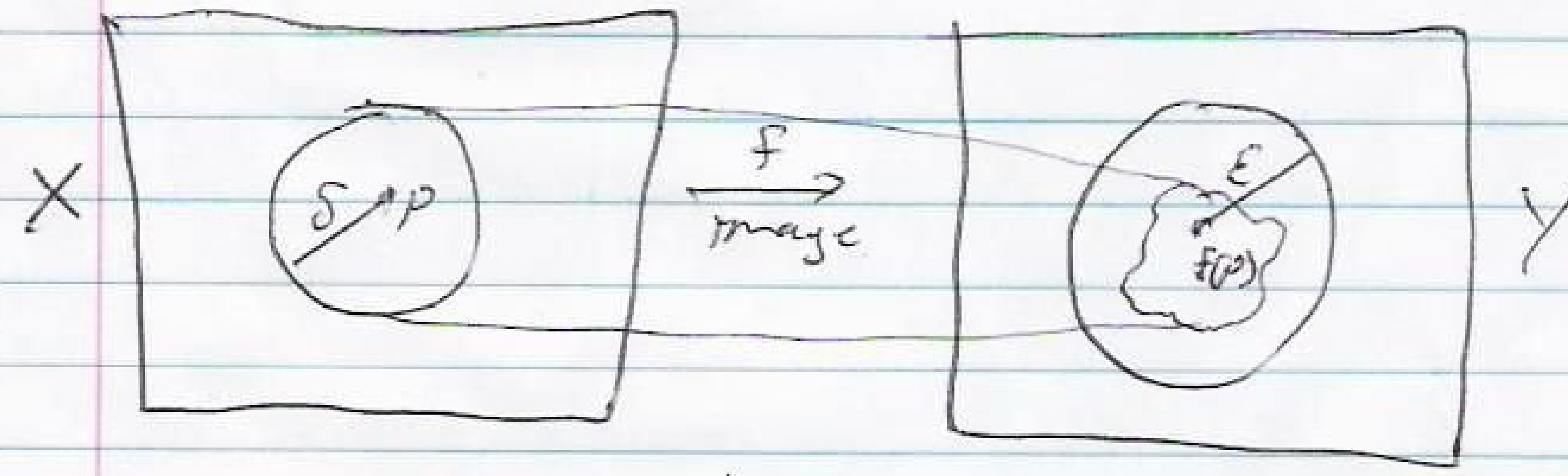
Let $f: X \to Y$ and (X, d_X), (Y, d_Y) be metric spaces. What is the ϵ-δ definition of continuity of f?

$$\forall p \in X \ \forall \epsilon \in \mathbb{R}_+ \ \exists \delta \in \mathbb{R}_+ \ f(B(p,\delta)) \subseteq B(f(p), \epsilon)$$

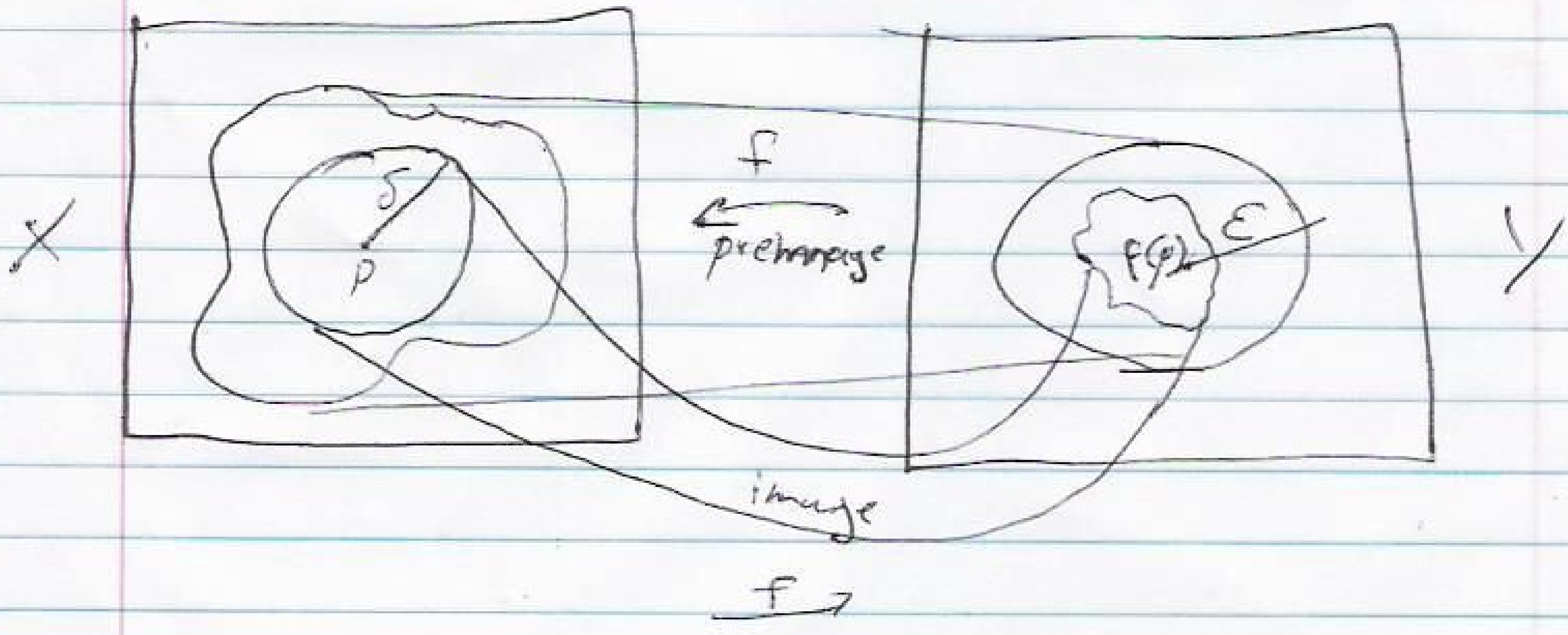
For every p in X, for every real $\epsilon > 0$, there exist a real $\delta > 0$ such that if $q \in X$ & $d_X(p,q) < \delta$, then $d_Y(p,q) < \epsilon$

d_y(f(p),f(q))<epsilon

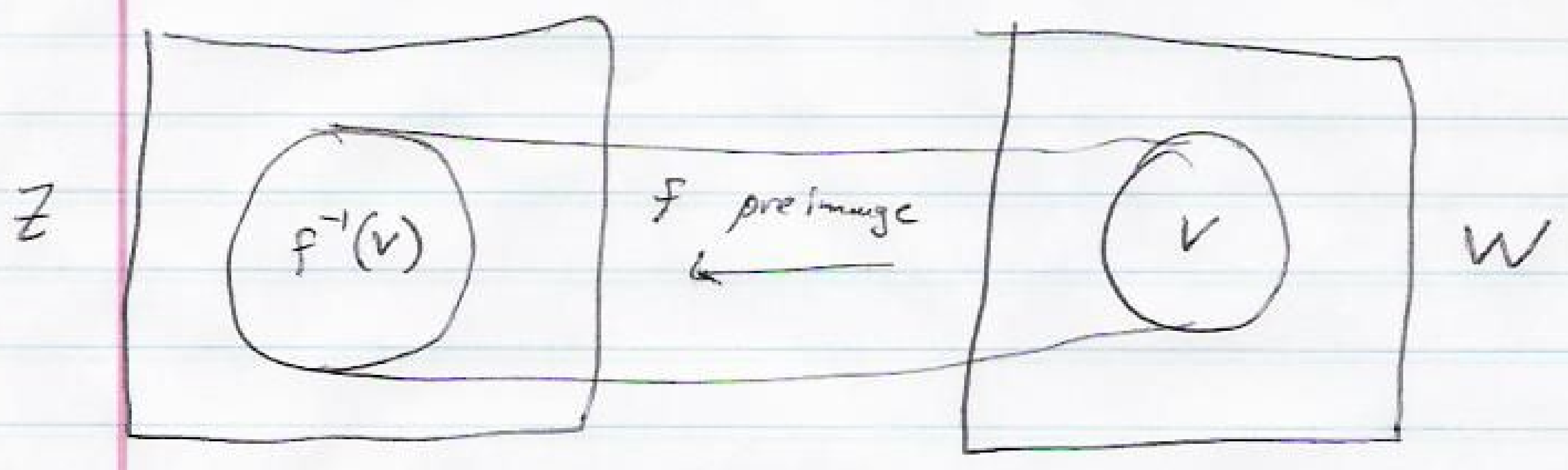
$$f(B(p,\delta)) \subset B(f(p), \epsilon)$$



Equivalently: $B(p,\delta) \subset f^{-1}(B(f(p), \epsilon))$



If Z, W topological spaces & $f: Z \to W$ then f is continuous $\iff \forall V$ open $\subset W$ $f^{-1}(V)$ open $\subset Z \iff \forall B \in \mathcal{B}$ $f^{-1}(U)$ open, where $\mathcal{B}$ is a basis for W's topology.



Recall $+ : \mathbb{R} \times \mathbb{R} \to$ is cts (with respect to the standard topologies on $\mathbb{R} \times \mathbb{R}$ & $\mathbb{R}$).

If X's topology is induced by the metric d_X, and Y's topology is induced by the metric d_Y, then continuity in the topological sense is equivalent to continuity in the $\varepsilon - \delta$ sense

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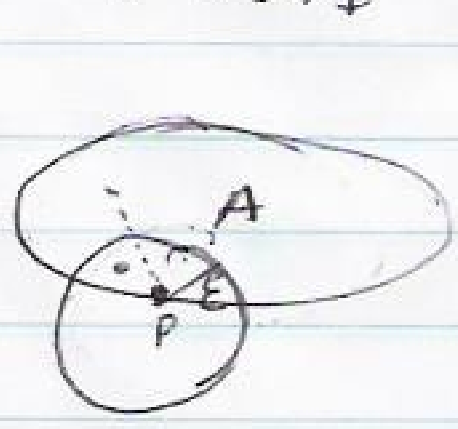
In a metric space, $p \in \bar{A} \iff$ $\exists (q_n)_{n \in \mathbb{Z}_+} \in A^\omega$ $(q_n)_{n \in \mathbb{Z}_+}$ converges to p.



$\iff \forall \varepsilon \in \mathbb{R}_+$ $B(p, \varepsilon) \cap A \neq \emptyset$

$\iff \forall n \in \mathbb{Z}_+$ $B(p, \frac{1}{n}) \cap A \neq \emptyset$

$\iff \forall n \in \mathbb{Z}_+$ $B(p, 2^{-n}) \cap A \neq \emptyset$ $\to$

$$\underbrace{B(p,\varepsilon)}_{\text{open ball of radius } \varepsilon \text{ and center } p} = \{ q \mid d(p,q) < \varepsilon \}$$

$$\underbrace{\overline{B(p,\varepsilon)}}_{\text{closed ball}} = \{ q \mid d(p,q) \le \varepsilon \}$$

True for $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$

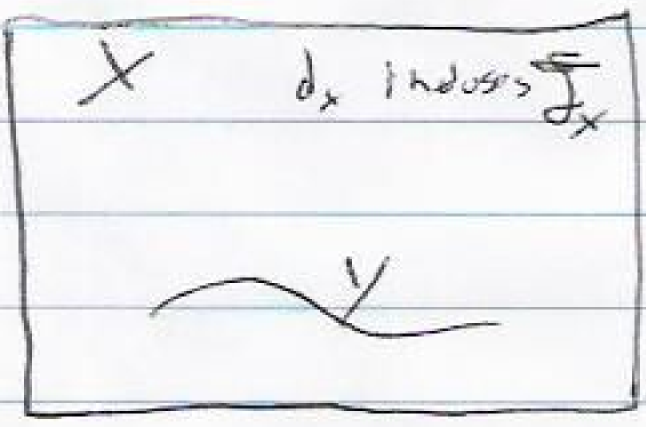
<u>not</u> true for

$X = [0,1] \cup [2,3]$

with $d(p,q) = |p-q|$

~~Submetrics induce the subspace topology~~

Restricting a metric to a subset induces the subspace topology.



$\{ U \cap Y \mid U \in \mathcal{T}_x \}$

d_x on Y induces $\mathcal{T}_Y$

$X = [0,1] \cup [2,3]$

$B(\frac{1}{2}, 1)$ in X

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$= \{ q \in X \mid |\tfrac{1}{2} - q| < 1 \} = [0,1]$

$\overline{B(\tfrac{1}{2},1)} = [0,1]$

because $B(\tfrac{1}{2},1)$ is closed. why?

$X - B(\tfrac{1}{2},1) = X - [0,1] = [2,3] = \overbrace{(\tfrac{3}{2}, \tfrac{7}{2})}^{\text{open in } \mathbb{R}} \cap X$

open in X

$\{ q \in X \mid |\tfrac{1}{2} - q| \leq 1 \} = \{ q \in X \mid |\tfrac{1}{2} - q| < 1 \} =$

$\shortparallel$

$[0,1]$ is closed

$\{ q \in X \mid |\tfrac{1}{2} - q| \leq \tfrac{1}{2} \}$

Moreover $B(\tfrac{1}{2},1) = \overline{B(\tfrac{1}{2},\tfrac{1}{2})}$ //

$\overline{B(\tfrac{1}{2},\tfrac{3}{2})} = \overline{\{ q \in X \mid |\tfrac{1}{2} - q| < \tfrac{3}{2} \}} = \overline{[0,1]} = [0,1]$

$\{ q \in X \mid |\tfrac{1}{2} - q| \leq \tfrac{3}{2} \} = [0,1] \cup \{2\}$