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Let $p:[0,1] \longrightarrow \{a,b,c\}$ where $p(0)=a$, $p(x)=b$ if $0<x<1$, and $p(1)=c$. What is the induced quotient topology on $\{a,b,c\}$?

$f: X \to Y$ is continuous if $\forall U \subset Y$ (U open $\Rightarrow f^{-1}(U)$ open)

$f: X \to Y$ is a quotient map if $\forall U \subset Y$ (U open $\Longleftrightarrow f^{-1}(U)$ open) and f is surjective.

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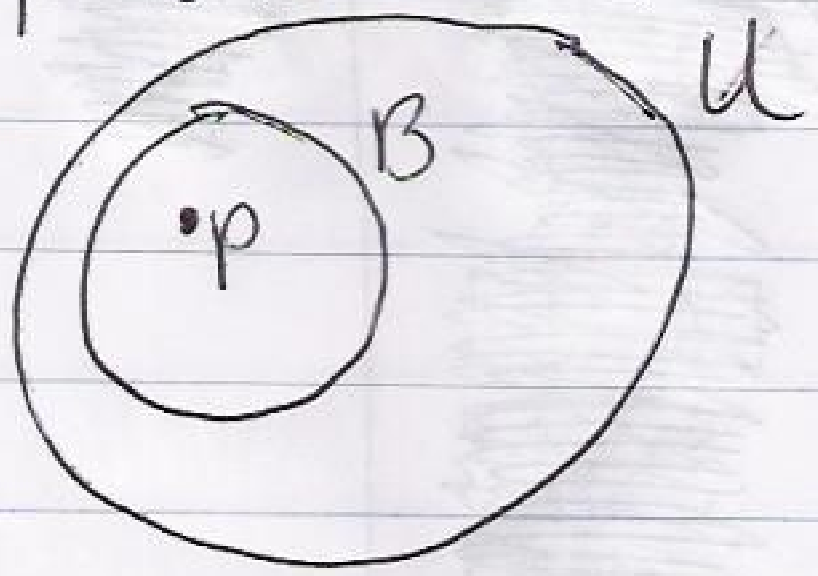
Informally, $\mathcal{U}$ is open in $[0,1]$ if and only if no point in $\mathcal{U}$ is infinitely close to $[0,1]-\mathcal{U}$.

$\mathcal{U} \subset \{a,b,c\}$

$\cancel{\emptyset}$	$\cancel{\emptyset}$	Yes
$\{a\}$	$\{0\}$	No: $0 \in \{0\}$ & 0 is infinetly to $[0,1]-\{0\}=(0,1]$
$\{b\}$	$(0,1)$	Yes: $(0,1)=(0,1)\cap[0,1]$ & $(0,1)$ is [open] in $\mathbb{R}$
$\{c\}$	$\{1\}$	No: $1 \in \{1\}$ & 1 is infinetly close to $[0,1)$
$\{a,b\}$	$[0,1)$	Yes: $[0,1)=(-1,1)\cap[0,1]$ & $(-1,1)$ is open in $\mathbb{R}$
$\{a,c\}$	$\{0,1\}$	No: $0 \in \{0,1\}$ 0 is infinetly close to $[0,1]-\{0,1\}=(0,1)$
$\{b,c\}$	$(0,1]$	Yes: $(0,1]=(0,2)\cap[0,1]$ & $(0,2)$ is open in $\mathbb{R}$
$\{a,b,c\}$	$[0,1]$	Yes

Formally, if $\mathcal{B}$ is a base at $[0,1]$ then $\mathcal{U}$ is open in $[0,1]$ if and only if $\forall p \in \mathcal{U} \, \exists B \in \mathcal{B}$
$p \in B \subset \mathcal{U}$



... however $[0,1]$ is metrizable.
E.g. $\inf d(p,1)=0$ $p \in [0,1)$ is the formal version of "1 is infinitly close to $[0,1)$."

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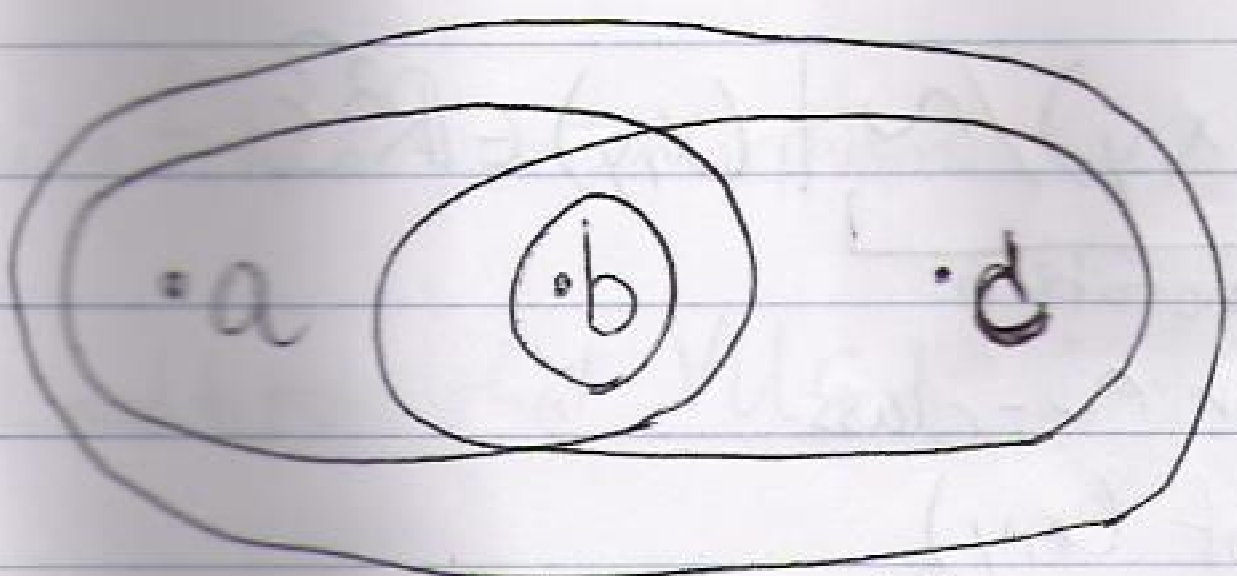

The quotient topology on Y induced by a surjection $f: X \longrightarrow Y$ is:

$\{U \subset Y \mid f^{-1}(U) \text{ is open in } X\}$

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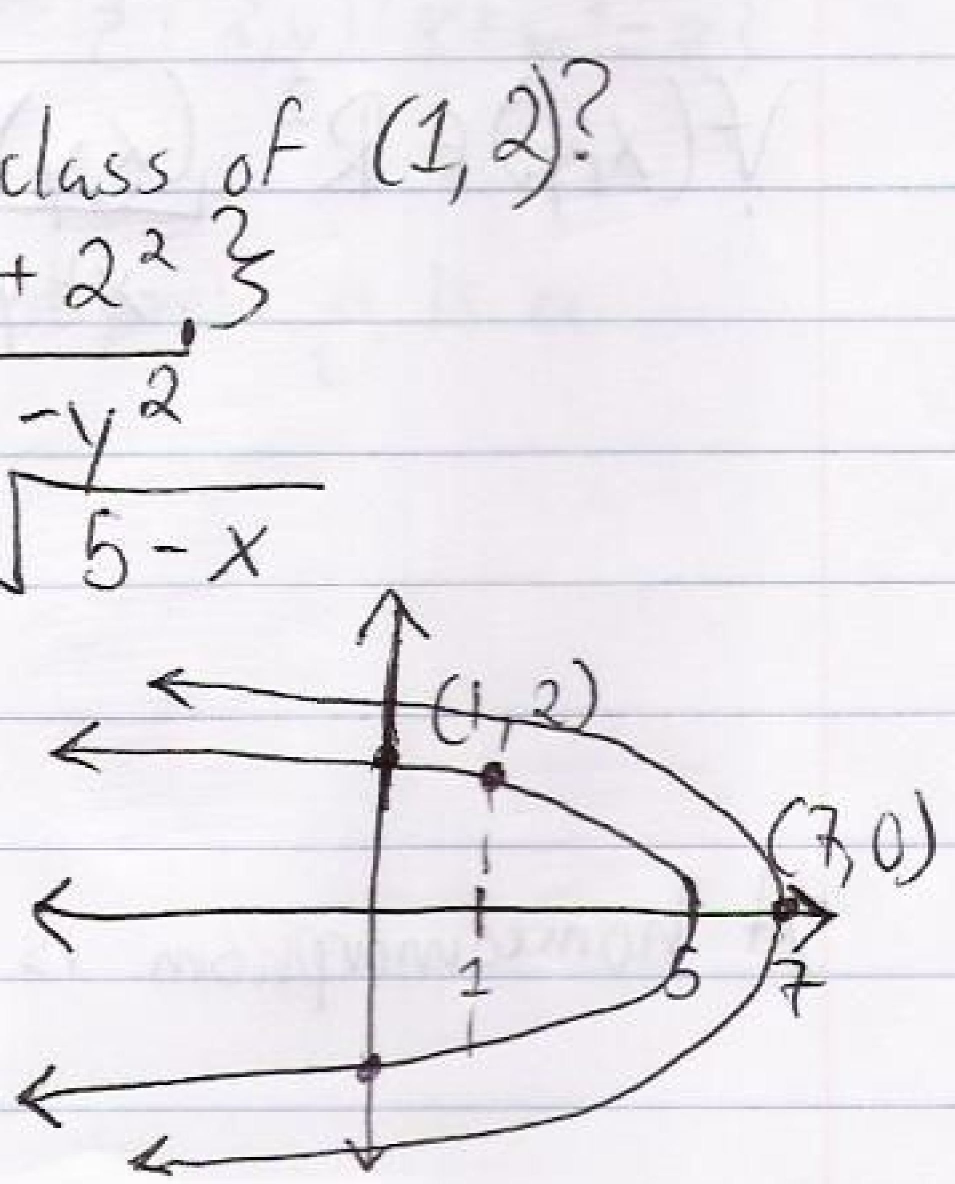
$(x_0, y_0) \sim (x_1, y_1) :\Longleftrightarrow x_0 + y_0^{\,2} = x_1 + y_1^{\,2}$

What is the eqivalence class of $(1, 2)$?

$\{(x,y) \in \mathbb{R}^2 \mid \underline{x + y^2 = 1 + 2^2}\}$

$\Longleftrightarrow \cancel{x} = 5 - y^2$

$\Longleftrightarrow y = \pm\sqrt{5 - x}$



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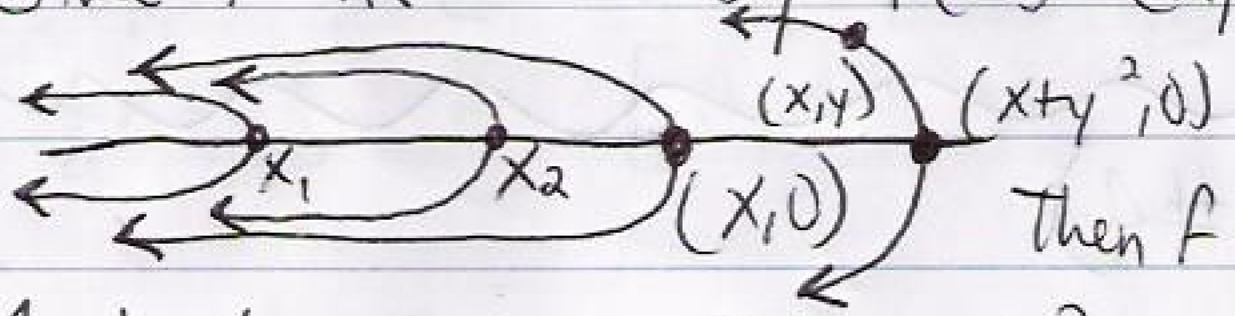
$$X^* = \{ \underbrace{(x,y)/\sim}_{\text{notation for } \sim\text{-class of } (x,y)} \mid (x,y) \in \mathbb{R}^2 \}.$$

The topology of X^* is induced by the surjection $p: \mathbb{R}^2 \longrightarrow X^*$ where $p(x,y) = (x,y)/\sim$

Define $f: \mathbb{R} \to X^*$ by $f(x) = (x,0)/\sim$.



Then f is bijective.

1-to-1: $x_0 \neq x_1 \Rightarrow x_0 + 0^2 \neq x_1 + 0^2$
$\Rightarrow (x_0,0)/\sim \; \neq (x_1,0)/\sim$

$\forall (x,y) \in \mathbb{R}^2 \; \underbrace{(x,y) \sim (x+y^2,0)}_{x+y^2 = (x+y^2)+0^2},$

so $(x,y)/\sim \; = (x+y^2,0)/\sim$
so $f(x+y^2)$
$= (x+y^2,0)/\sim$
$= (x,y)/\sim$

A homeomorphism is a bijective quotient map.

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Claim f is a homeomorphism.

We need to prove that $\forall U \subset X^*$ U is open $\Longleftrightarrow f^{-1}(U)$ is open

Hint: $g(x,y) = x+y^2$

[redacted]

$$\begin{tikzcd}
(x,y) \in \mathbb{R}^2 \arrow[r, "p"] \arrow[dr, "g"'] & (x,y)/\sim \, \in X^* \\
 & x+y^2 \in \mathbb{R} = Z \arrow[u, "f"']
\end{tikzcd}$$

continuous? ✓

surjective? ✓

$\boxed{\text{Cor. } 22.3}$

$g(r,0) = r$

$g^{-1}(\{x+y^2\}) = (x,y)/\sim$

$g^{-1}(\{z\}) = \{(x,y) \mid x+y^2 = z\}$

By Cor. 22.3, we just need to prove g is a quotient map!

Read Section 23 8