Topology Notes

§24 Connected Subspaces of the Real line

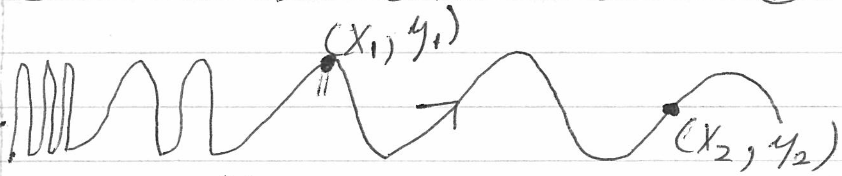
Quiz: Give an example of a connected, but not path connected space.

① Recall: If $A \subset X$
and A connected
" $A \subset B \subset \bar{A}$,
then B is connected too

② Path-connected $\Rightarrow$ connected.

$S = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y = (\frac{1}{x})\}$

② $\Rightarrow S$ is connected $\Rightarrow$ ① $\bar{S}$ is connected.



$0 \leftarrow x$ $\quad x = 0 \quad -1 \leq y \leq 1$

$\bar{S} = S \cup (\{0\} \times [-1, 1])$

I claim S is path connected.

③ If $f_1 : X \to Y_1$, and $f_2 : X \to Y_2$
are cts, then $f_1 \times f_2 : X \to Y_1 \times Y_2$
is continuous too.

Suppose $(x_1, y_1), (x_2, y_2) \in S$.
Assume $x_1 \leq x_2$.
The other case is handed symetrically

continuous $f : [x_1, x_2] \to [x_1, x_2]$ $\quad \underbrace{x \mapsto x}_{\text{identity group}}$

$g : [x_1, x_2] \to \mathbb{R}$

$x \mapsto \sin(\frac{1}{x})$

$f \times g : [x_1, x_2] \to [x_1, x_2] \times \mathbb{R}$

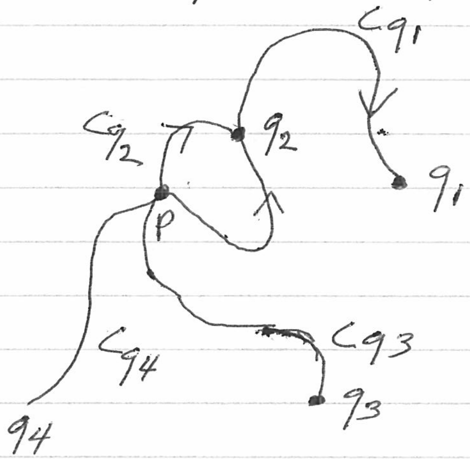
$x \mapsto (x, \sin(1/x))$

f, g cts. $\Rightarrow f \times g$ cts.

Actually, $f \times g$ maps into S
because $(x, \sin(1/x)) \in S \ \forall \ x > 0$

Proof of 2

Assume X is path connected
choose $p \in X$. For every $q \in X$, choose a path C_q from p to q.



Then $X = \bigcup_{q \in X} C_q$

3.5 Intervals are connected

4 Continuous surjections preserve connectedness

5 A union of connected subspaces $\bigcup_{i \in I} K_i$ is connected if $\forall$ K's have a common point.

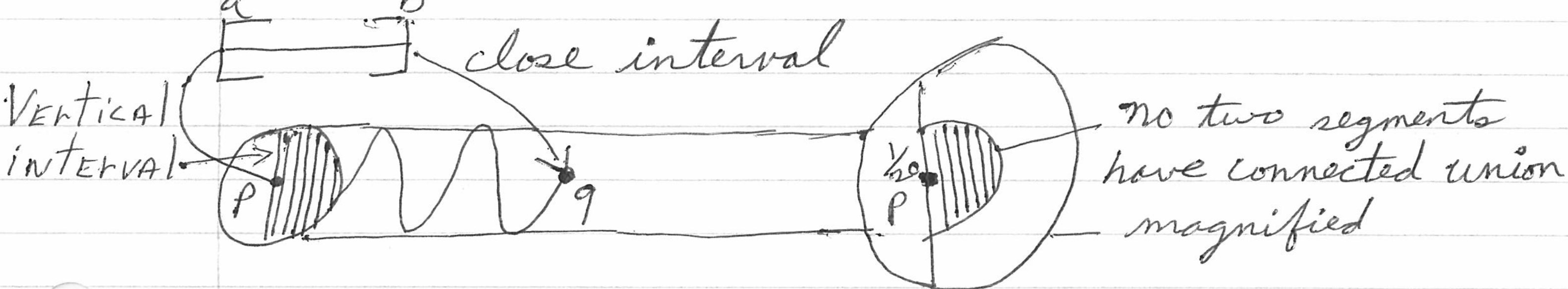
By definition of path, every C_q is a cts. image of a closed interval. By facts 3.5 and 4 every C_q is connected.
All the C_q's contain p, so X is connected by 5. $\square$

$p = (0,0) \in \overline{S}$

$q = \left(\frac{1}{\pi}, 0\right) \in \overline{S} \longrightarrow 0 = \sin\left(\frac{1}{1/\pi}\right)$

Claim: There is no path from p to q in $\overline{S}$.

Suppose $[a,b] \to \overline{S}$
and $f(a) = p$ and $f(b) = q$.
I claim f is not continuous.



We want to prove this
Consider the ball of radius $\frac{1}{20}$ around p.
$B\left(p, \frac{1}{20}\right)$

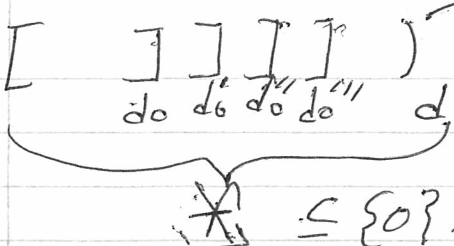
Consider the preimage $\mathcal{U} = f^{-1}\left(B\left(p, \frac{1}{20}\right)\right)$.

If $\mathcal{U}$ is not open, then f is not cts.
Case "$\mathcal{U}$ is open": $f(a) = p \in B\left(p, \frac{1}{20}\right)$
$\Rightarrow a \in f^{-1}\left(B\left(p, \frac{1}{20}\right)\right) = \mathcal{U}$.

$\Rightarrow [a, c) \subset \mathcal{U}$ for some $c > a$.
$[a, c)$ is connected $\Rightarrow f([a,c))$ is connected if f is continuous. So, assuming $f(a,c))$ is connected, $f([a,c)) \subseteq \{0\} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$

Let d be the supremum of $d_0 \leq b$ where $f([a, d_0]) \subseteq \{0\} \times [-1, 1]$

This is saying $f([a,d)) \subseteq \{0\} \times [-1,1]$



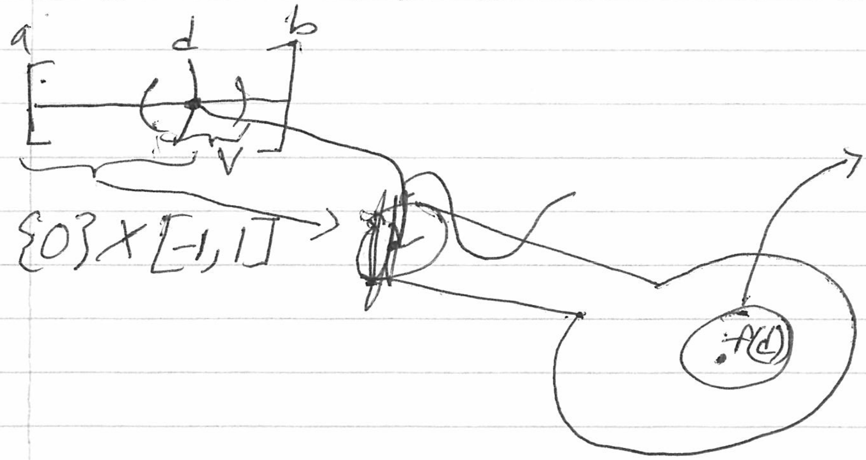
$\subseteq \{0\} \times [-1,1]$

If $d < b$ and $f([a,d]) \subseteq \{0\} \times [-1,1]$, then we can repeat our earlier argument for a to show $f([d,d')) \subseteq \{0\} \times [-1,1]$ for some $d' > d$. But in this case, $f([a,d)) \subseteq \{0\} \times [-1,1]$.

Conclusion: $d = b$ and $f([a,b]) \subseteq \{0\} \times [-1,1]$ or $d < b$ and $f(d) \notin \{0\} \times [-1,1]$.

In the first case $f(b) \neq q$. $\Rightarrow \Leftarrow$.

2nd case: We conclude f is not continuous.



Consider $B(f(d)), \varepsilon)$ for ε small enough

$V = f^{-1}(B(f(d), \varepsilon))$.

f cts $\Rightarrow$ V open $\Rightarrow$ $(h,k) \subset V$ for some h, k with $a \leq h < k \leq b$.

But if $h < x < d$, then $x \notin V$ because $f(x) \in \{0\} \times [-1,1]$, so $f(x) \notin B(f(d), \varepsilon)$ so $x \notin f^{-1}(B(f(d), \varepsilon)) = V$

next section 25, note warm-up?