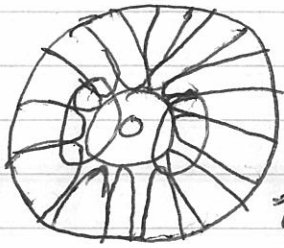
MATH 4360
Nov. 16, 2010

§26 Compact Spaces

Define compactness: A space $(X, \mathcal{T})$ is compact if $\forall \mathcal{U} \subset \mathcal{T} \ (\bigcup \mathcal{U} = X \Rightarrow \exists \mathcal{F}$ finite $\subset \mathcal{U} \ \bigcup \mathcal{F} = X)$

$\forall$ [families $\mathcal{U}$ of opens sets] if $\mathcal{U}$ covers X, then $\mathcal{U}$ has a finite subset that covers X, every open cover has a finite subcover.



a bunch of open sets

$X = \bigcup \mathcal{U}$

$\mathcal{U}$ is a family of open sets

$\mathcal{U}$ is also a cover

$X = [0,1]$

$$\mathcal{U} = \{\underbrace{[0, 1-\tfrac{1}{n})}_{= (-1,\, 1-\frac{1}{n}) \cap X} : n \in \mathbb{Z}_+\} \cup \{\underbrace{(0.84, 1]}_{= (0.84,\, 2) \cap X}\}$$

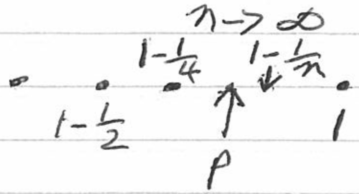
Proof: $\mathcal{U}$ covers X

$\forall p \in [0,1)$

$\exists n$ big enough that

$p < 1 - \frac{1}{n}$ (so $p \in [0, 1-\frac{1}{n}) \in \mathcal{U}$)

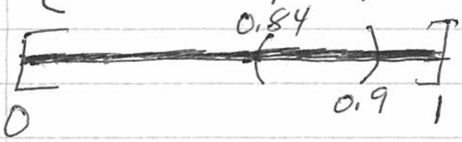
because $\lim_{n \to \infty} (1 - \frac{1}{n}) = 1 - 0 = 1$



So, $[0,1) \subset \bigcup \mathcal{U}$

$1 \in (0.84, 1]$, so $X = [0,1) \cup \{1\} \subset \cup \mathcal{U}$.
So yes, $\mathcal{U}$ covers X.

Open cover is made up of a family of open sets
Since every $U \in \mathcal{U}$ is open, $\mathcal{U}$ is an open cover of X.
$\{[0. 1-\frac{1}{10}), (0.84, 1]\}$ is a finite subcover of $\mathcal{U}$.



everything is covered

Suppose X is finite
Then X is compact
Proof: Let X have n points.
Let $\mathcal{U}$ be an open cover of X. Every $V \in \mathcal{U}$ is a subset of X.
But there are only 2^n subsets of X. So, $\mathcal{U}$ is a finite subcover of $\mathcal{U}$, for $\mathcal{U}$ has at most 2^n elements.
. . . . three independent choices

Warm up? $[0,5] \cap Q$ is not compact
Proof: We just need to find an open cover with no finite subcover.

Informal definition of compactness, X is compact if X has no holes.
To proof not compact find a <u>hole</u>

$X = [0,5] \cap Q$
$\sqrt{2} \in \mathbb{R} - X$. Lets use this "hole" at $\sqrt{2}$.

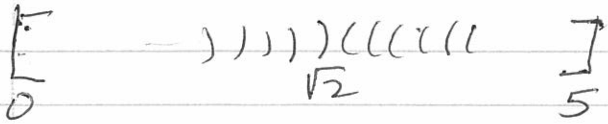
$$\mathcal{U}_n = \underbrace{\left((-1, \sqrt{2} - \tfrac{1}{n}) \cup (\sqrt{2} + \tfrac{1}{n}, 6)\right)}_{\text{open in } \mathbb{R}} \cap X \text{ is open in } X.$$

$\mathcal{U} = \{ \mathcal{U}_n : n \in \mathbb{Z}_+ \}$ is an open cover of X.

If $p \in X \Rightarrow p \neq \sqrt{2} \Rightarrow \begin{cases} p \in [0, \sqrt{2}) \cap X \Rightarrow \exists n \in \mathbb{Z}_+ \\ \text{or} \\ p \in (\sqrt{2}, 5] \cap X \Rightarrow \exists n \in \mathbb{Z}_+ \end{cases}$

$p \in (-1, \sqrt{2} - \frac{1}{n}) \cap X$

$p \in (\sqrt{2} + \frac{1}{n}, 6) \cap X$



$\mathcal{U}$ has no finite subcover

If $\mathcal{F}$ finite $\subset \mathcal{U}$, then

$\mathcal{F} \subset \{ \mathcal{U}_1, \dots, \mathcal{U}_N \}$ for

some N. $\cup \mathcal{F} \in \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_N$

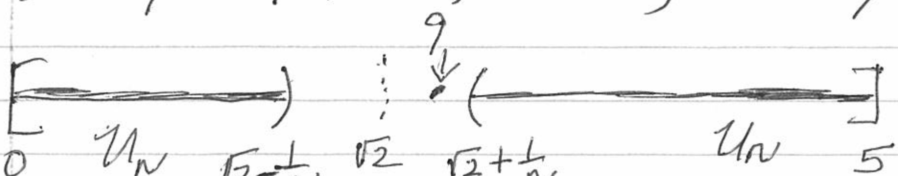
$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset \dots \subset \mathcal{U}_N \Rightarrow \mathcal{U}_1 \cup \dots \mathcal{U}_N = \mathcal{U}_N$

So, $\cup \mathcal{F} \subset \mathcal{U}_N$

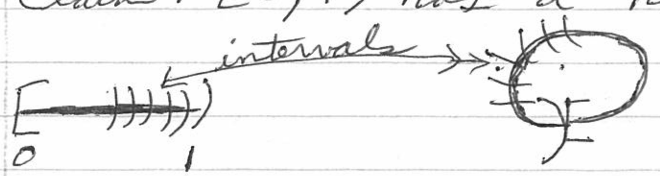
Fact: Between any two reals is a rational,

so $\exists q \in \mathbb{Q}$ $\sqrt{2} < q < \sqrt{2} + \frac{1}{N}$

so $q \in \mathbb{Q} \cap [0,5] = X$, but $q \notin \mathcal{U}_N$



Claim: $[0,1)$ has a "hole" at 1

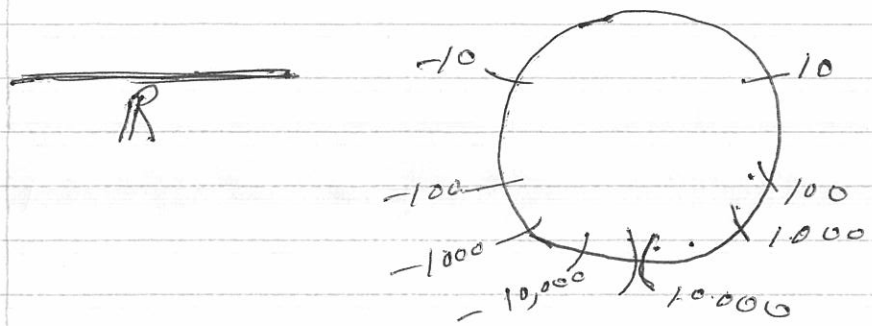


$\mathcal{U} = \{ [0, 1 - \frac{1}{n}) : n \in \mathbb{Z}_+ \}$

is an open cover of $[0,1]$

with no finite subcover.

$\mathbb{R}$ has a "hole at ∞"



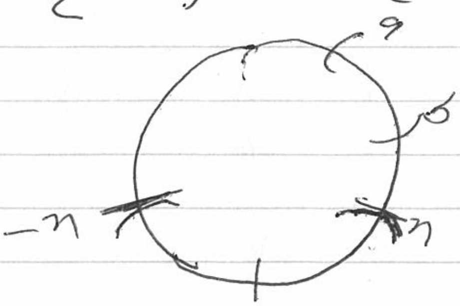
$\mathcal{U} = \{(-n,n) \mid n \in \mathbb{Z}_+\}$
is an open cover of $\mathbb{R}$ with no finite subcover.

Define a topology $\mathcal{T}$ on $\mathbb{R} \cup \{\infty\}$ by declaring $\mathcal{B}$ to be a basis of $\mathcal{T}$, where

copy of circle

$\mathcal{B} = \{(a,b) \mid a < b \in \mathbb{R}\}$

$\cup \{(n,\infty) \cup \{\infty\} \cup (-\infty,-n)$ where $n \in \mathbb{Z}_+\}$



∞ ← sets in $\mathcal{B}$

• holes are limit pts in compact spaces
~~limit pts are holes~~

$\mathcal{T} = \{\cup \mathcal{E} \mid \mathcal{E} \subset \mathcal{B}\}$

~~Theorem~~ Continuous images (Theorem 26.5), closed subspaces (Theorem 26.2), and products (Theorem 26.7) all preserve compactness.

Theorem (26.3) If X is Hausdorff and K compact subspace of X) $\Rightarrow$ K is closed in X.

• "limit pts are holes in Hausdorff spaces"

Hausdorff $\Leftrightarrow \forall p, q \in X \; (p \neq q \Rightarrow \exists\, \mathcal{U}, V$ open

$(\cdot p)^{\mathcal{U}}$ $(\cdot q)^{V}$ $p \in \mathcal{U}$, $q \in V$, and $\mathcal{U} \cap V = \emptyset$