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Topology MATH 4360

Notes for 11-23-10

The answer for the quiz: $\mathcal{B} = \{(a,b) \mid a<b,\ a,b \in \mathbb{Q}\}$.

Quiz Question:

Give an example of a countable basis of $\mathbb{R}$.

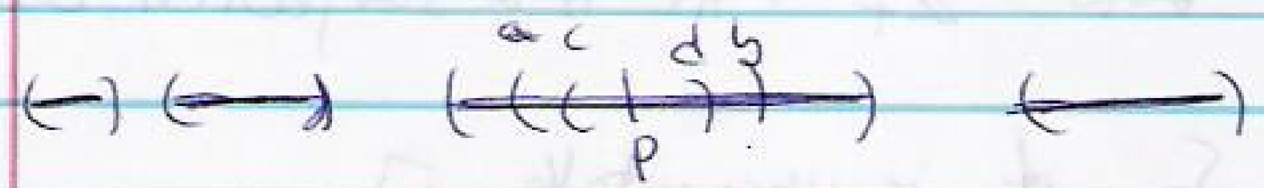
"$\mathcal{B}$ is a basis for (the standard topology of) $\mathbb{R}$" means:

$\forall\, \mathcal{U}$ open $\subset \mathbb{R}$ $\forall p \in \mathcal{U}$ $\exists B \in \mathcal{B}$ $p \in B \subset \mathcal{U}$.

By definition $\mathcal{A} = \{(a,b) \mid a<b,\ a,b \in \mathbb{R}\}$ is a basis for $\mathbb{R}$.

That means $\forall\, \mathcal{U} \subset \mathbb{R}$ $[\mathcal{U}$ is open $\Leftrightarrow \forall p \in \mathcal{U}$ $\exists B \in \mathcal{A}$ $p \in B \subset \mathcal{U}]$

Suppose $\mathcal{U}$ open $\subset \mathbb{R}$ and $p \in \mathcal{U}$. Then $p \in (a,b) \subset \mathcal{U}$ for some $a, b \in \mathbb{R}$.




$\mathcal{U}$. Choose $c, d \in \mathbb{Q}$ such that $a<c<p<d<b$. Then $(c,d) \in \mathcal{B}$ and $p \in (c,d) \subset \mathcal{U}$.

Therefore, $\forall\, \mathcal{U}$ open $\subset \mathbb{R}$ $\forall p \in \mathcal{U}$ $\exists B \in \mathcal{B}$ $p \in B \subset \mathcal{U}$.

So, $\mathcal{B}$ is a basis for $\mathbb{R}$ with the standard topology.

$\mathbb{Q}$ is countable, i.e., there is a surjection $f: \mathbb{Z}_+ \to \mathbb{Q}$.

$\mathbb{R}$ is not countable, i.e., there is no surjection $f: \mathbb{Z}_+ \to \mathbb{R}$
↑ Corollary 27.2

$\mathbb{Q}$ countable $\Rightarrow \mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ is countable
$\Rightarrow \exists$ surjection $g: \mathbb{Z}_+ \to \mathbb{Q}^2$

Let $h : \mathbb{Q}^2 \to \mathcal{B} = \{(a,b) \mid a<b,\ a,b \in \mathbb{Q}\}$ where

$$h(a,b) = \begin{cases} (a,b) : a<b. \\ (0,1) : a \geq b. \end{cases}$$

$\mathbb{Z}_+ \xrightarrow{g} \mathbb{Q}^2 \xrightarrow{h} \mathcal{B}$. So, $\overset{h\circ g}{\cancel{g\circ h}} : \mathbb{Z}_+ \to \mathcal{B}$ is a surjection

because $g$ and $h$ are surjections. So, $\mathcal{B}$ is ctbl.

$\mathcal{A} = \{(a,b) \mid a<b,\ a,b \in \mathbb{R}\}$ $\mathcal{A}$ is uncountable

Proof: Suppose not. Then $\exists v : \mathbb{Z}_+ \to \mathcal{A}$ a surjection.

Let $w : \mathcal{A} \to \mathbb{R}$ where $w((a,b)) = a$. Then $w$ is a

surjection. So, $\overset{w\circ v}{\cancel{v\circ w}}$ $\mathbb{Z}_+ \to \mathbb{R}$ is a surjection, contradicting

Corollary 27.8. So, $\mathcal{A}$ is uncountable. $\square$

Now for the warm-up question: But first a definition.

A subset $A$ of a space $X$ is dense if $\bar{A} = X$. Equivalently,

$\forall p \in X$ $p \in A$ or $\underline{p \text{ is a limit point of } A}$,

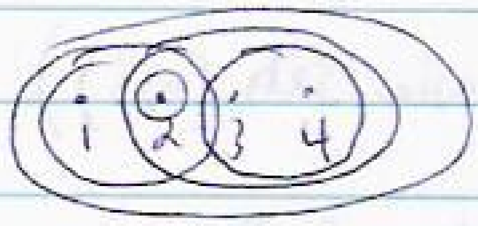
$\Leftrightarrow \forall U$ open $[p \in U \Rightarrow U \cap A \neq \emptyset]$

$\Leftrightarrow \forall (U \text{ nhbd of } p)\ U \cap A \neq \emptyset$

Equivalently, $\forall U$ open $\neq \emptyset$ $U \cap A \neq \emptyset$.

$\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$: $\forall p \in \mathbb{R}, \forall U$ nhbd

of $p$, $\exists (a,b) \subset U$ $\exists q \in \mathbb{Q} \cap (a,b) \subseteq \mathbb{Q} \cap U$



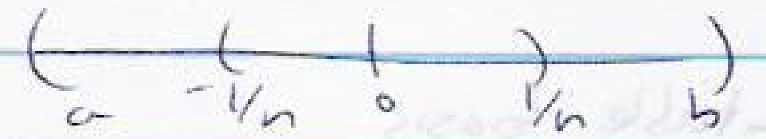

$\overline{\{1,2\}} = \{1,2\} \Rightarrow \{1,2\}$ not dense in $\{1,2,3,4\}$
$\overline{\{1,4\}} = \{1,3,4\} \Rightarrow \{1,4\}$ not dense
$\overline{\{2,3\}} = \{1,2,3,4\} \Rightarrow \{2,3\}$ is dense
$\overline{\{2,4\}} = \{1,2,3,4\} \Rightarrow \{2,4\}$ is dense

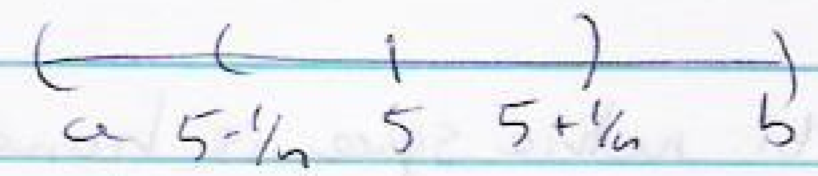
These are the answers for the warm-up solution.

$\mathcal{B}$ is a basis at a point $p$ in a space $X$ if $\forall\, U$ open

$[p \in U \Rightarrow \exists B \in \mathcal{B},\ p \in B \subset U]$.

Consider $\{(-1/n, 1/n) : n \in \mathbb{Z}_+\}$ is a basis $0$ in $\mathbb{R}$.

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Separable $\Leftrightarrow$ has ctbl dense set.

1st ctbl $\Leftrightarrow$ $\forall p\ \exists$ ctbl basis at $p$

$\Uparrow$

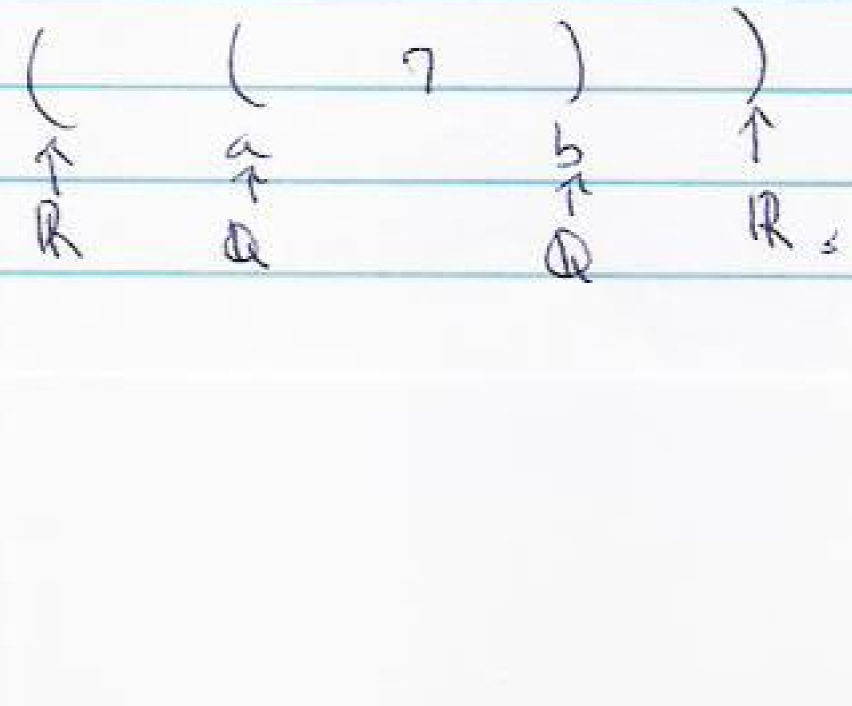
2nd ctbl $\Leftrightarrow$ $\exists$ ctbl basis

| $\mathbb{R}$ | All metric spaces... |
|---|---|
| yes. $\mathbb{Q}$ | ? |
| yes | yes |
| Yes. rational intervals | ~~rational intervals~~ ? |

$\mathcal{B} = \{(a,b) \mid a<b \quad a,b \in \mathbb{Q}\}$ is a ctbl. basis of $\mathbb{R}$.

$\mathcal{C} = \{(a,b) \mid a<7<b,\ a,b \in \mathbb{Q}\} \subset \mathcal{B}$ is a ctbl

basis at 7.



| | $\mathbb{R}$ with discrete topology | $\mathbb{R}_\ell$ (basic sets $[a,b)$) |
|---|---|---|
| separable $\Leftrightarrow$ ctbl dense set | no: $\mathbb{R}$ is only dense set | yes: $\mathbb{Q}$ |
| 1st ctbl $\Leftrightarrow$ ... | yes: $\{\{p\}\}$ basis at $p$ | yes: $\{[p, p+1/n) : n \in \mathbb{Z}_+\}$ |
| 2nd ctbl $\Leftrightarrow$ ... | no | no |

Every compact metric space is 2nd ctbl.:

$\forall n \in \mathbb{Z}_+$, let $\mathcal{F}_n$ be a finite subcover of the set of all balls of radius $1/n$.

$\bigcup_{n \in \mathbb{Z}_+} \mathcal{F}_n$ is a countable basis.

Every 2nd ctbl. metric space is homeomorphic to a subspace of $\mathbb{R}^\omega \leftarrow \{ f \mid f : \mathbb{Z}_+ \to \mathbb{R} \} = \prod_{n \in \mathbb{Z}_+} \mathbb{R}$.