11/30/10

Big picture

① distance (metrics) vs. infinite closeness
(topology)

$d(p, B) = \inf \{ d(p,q) \mid p \in B,$

Compare to $p \in \overline{B}$

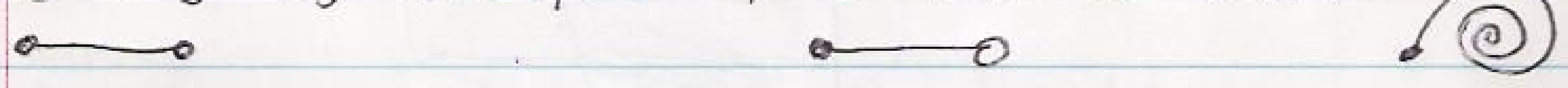
In the metric topology, $p \in \overline{B}$ $d(p,B)=0$

② Connectedness : all in "one piece"

$[0,1)$ is connected; $[0,1) \cup (1,2]$ is not.

③ compactness : "no holes"

$[0,1]$ is compact ; $[0,1)$ is not.



More concepts

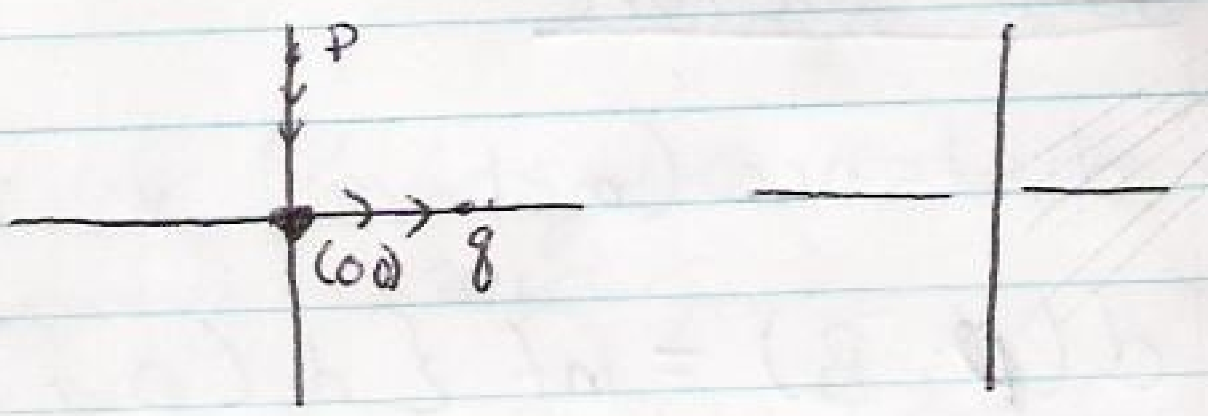
④ countability
⑤ Housdorff spaces
⑥ continuous function

Key tools

- Bases !
- neighborhoods
- limit points
- Closure
- subspace topology
- product topology
- quotient top.
- metric top.

Ex 1 Practice Questions

$X = \{(x,y) \in \mathbb{R}^2 / x=0 \text{ or } y=0\}$



Ex #6

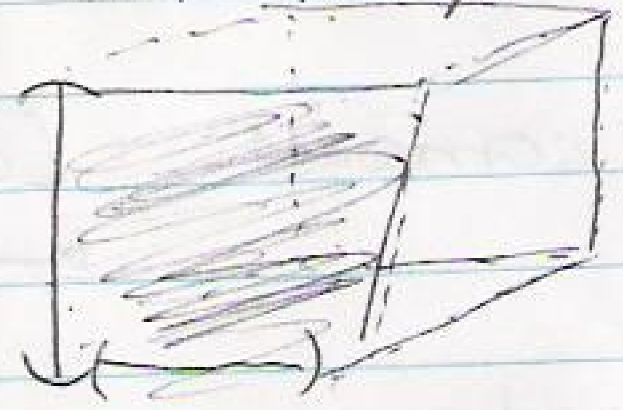
$\{(a,b) \mid a<b \text{ and } a,b \in \mathbb{R}\} = \mathcal{B}$ is a basis

for (the standard topology of) $\mathbb{R}$ (by definition)

Therefore $\{U \times V \times W \mid U, V, W \in \mathcal{B}\} = \mathcal{B}_\pi$

is a basis for the product topology

of $\mathbb{R}^3$



$\pi_1 : \mathbb{R}^3 \to \mathbb{R} \quad \pi_1(x_1, x_2, x_3) = x_1$

$\pi_2 : \mathbb{R}^3 \to \mathbb{R} \quad \pi_2(x_1, x_2, x_3) = x_2$

$\pi_1 : \mathbb{R}^3 \to \mathbb{R}$ is continuous if and

only if for all $U \in \mathcal{B}$, $\pi^{-1} U$ is open.

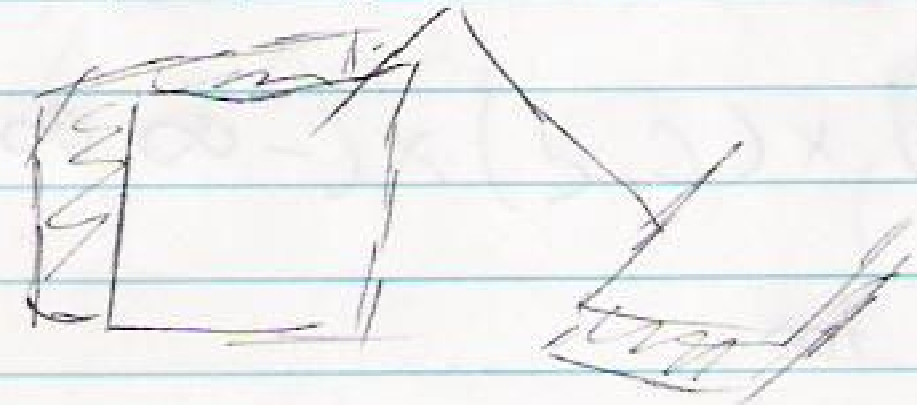
E.g. $\pi_1^{-1}(5,7) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 5 < x_1 < 7\}$

$\mathcal{J}_{12}$ is the smallest topology π_1 and π_2

are continuous.

So $\mathcal{J}_{12}$ is the smallest topology including all sets of the form $\pi_1^{-1}(a,b)$ and $\pi_2^{-1}(a,b)$

Thes sets are unions of open boxes



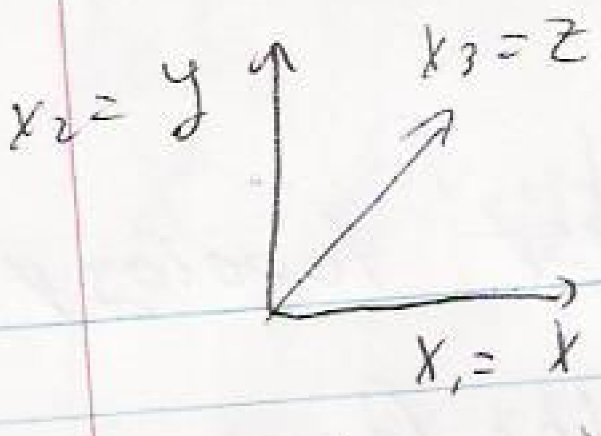
Eg: $\pi^{-1}(2,3) = \bigcup_{n\in\mathbb{Z}_+}\left((2,3)\times(-n,n)\times(-n,n)\right)$

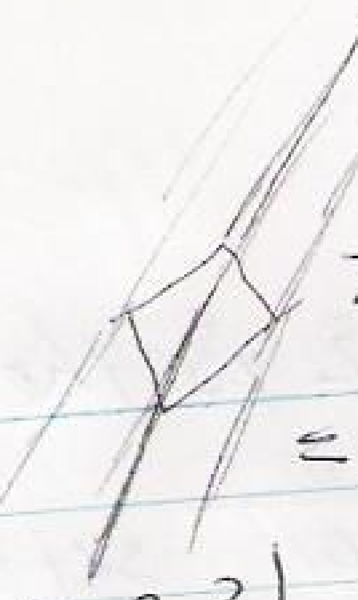
So, $\pi^{-1}(a,b)$ and $\pi_2(a,b)$ are in the product topology. So $\mathcal{J}_{12}$ is smaller then or equal to the product topology.

Any topology $\mathcal{J}$ on $\mathbb{R}^3$ must include $\emptyset$ and $\mathbb{R}^3$ as elemets and for all finite $\{U_1, \ldots U_n\} \subseteq \mathcal{J}$, $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{J}$ and for all $\mathcal{S} \subseteq \mathcal{J}$, $\bigcup_{U\in\mathcal{S}} U \in \mathcal{J}$.

Because of the phenomenon where for example $\left(\bigcup_{U\in\mathcal{S}_1} U\right) \cap \left(\bigcup_{V\in\mathcal{S}_2} V\right) = \bigcup_{(U,V)\in\mathcal{S}_1\times\mathcal{S}_2} (U\cap V)$

$\mathcal{J}_{12} = \{\bigcup_{U\in\mathcal{S}} U \mid \mathcal{S} \subset \{\pi_1^{-1}(a,b) \cap \pi_2^{-1}(c,d) \mid a<b \text{ and } c<d \text{ \& } a,b,c,d \in \mathbb{R}\}\}$





E.g.

$\pi_1^{-1}(3,4) \cap \pi_2^{-1}(5,6)$

$= (3,4) \times (5,6) \times (-\infty, \infty)$

$p = (1,2,3)$ $\quad$ $q = (1,2,3)$

then $p \in (a,b) \times (c,d) \times (-\infty, \infty) \subseteq U$.

So $a < 1 < b$ and $c < 2 < d$.

so $q \in (a,b) \times (c,d) \times (-\infty\ \infty) \subseteq U$,

so $q \in U$ too.

But $p \in (0,2) \times (1,3) \times (2,4)$

and $q \in (0,2) \times (1,3) \times (2,4)$

$2 < 4 < 4$ is false

So, $(0\ 2) \times (1,3) \times (2,4) \notin \mathcal{T}_{12}$.

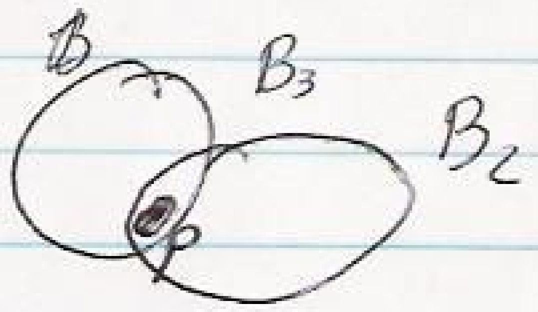
and $(0,2) \times (1,3) \times (2,4)$ is in the product topology.

(B) Zarisk topology of $\mathbb{R}$

Basis open sets are of the form $\emptyset$ $\mathbb{R}$, or $\mathbb{R} - \{r_1, \ldots, r_n\} = (-\infty, r) \cup (r_1, r_2) \cup \ldots \cup (r_n, \infty)$ if $r_1 < r_2 < \ldots < r_n$!

Basis properties $\mathcal{B}$ is a basis for some topology on X, if

① $\forall B \in \mathcal{B}$ $B \subseteq X$
② $\forall p \in X$ $\exists B \in \mathcal{B}$ $p \in B$
③ $\forall B_1, B_2 \in \mathcal{B}$ $\forall p \in B \cap B_2$ $\exists B_3 \in \mathcal{B}$
$p \in B_3 \subseteq B \cap B_2$



$\mathcal{B}$ is a basis of a topology $\mathcal{T}$.

means: several equivalent things

① $\mathcal{T}$ is the smallest topology containing $\mathcal{B}$

and $\mathcal{B}$ is a basis for some topology"

② $\mathcal{T} = \{\bigcup_{U \in \mathcal{S}} U \mid \mathcal{S} \subseteq \mathcal{B}\}$

③ $\mathcal{T} = \{U \mid \forall p \in U \ \exists B \in \mathcal{B} \ p \in B \subseteq U\}$