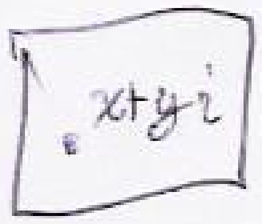
Dec. 2, 2010

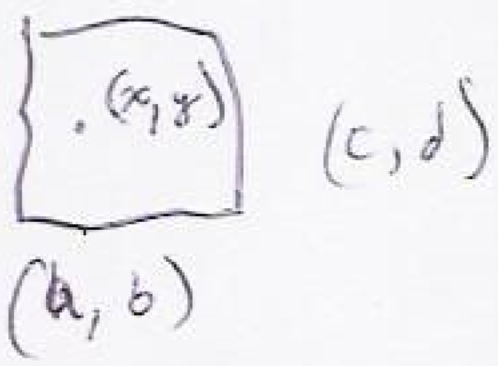
$\mathbb{C} \cong \mathbb{R}^2$

$x + yi \longrightarrow (x, y)$

Basis

$\{(a,b) \times (c,d) \mid a < b \ \& \ c < d\}$





$\uparrow q$

$\{x + yi \mid a < x < b \ \& \ c < y < d\}$

$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = S \subseteq \mathbb{R}^2$

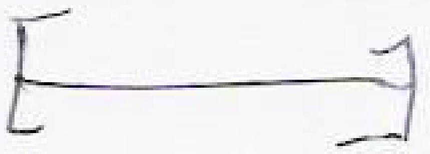
$x + yi = z \longmapsto e^z = e^{x + yi} = e^x(\cos y + i \sin y)$

$(x, y) \longmapsto (e^x \cos y, \ e^x \sin y)$

$x \xmapsto{f} e^{2\pi i x} = e^{0 + (2\pi x)i} = e^0 e^{2\pi x i}$

$x \xmapsto{f} (\cos 2\pi x, \ \sin 2\pi x)$

$T = [0, 1]/\sim$ where $x \sim y \iff [x = y \text{ or } \{x, y\} = \{0, 1\}]$



$[0, 1]$

T

$f : T \to S$

$\cos^2 2\pi x + \sin^2 2\pi x = 1$

$f([x]) = f(x/\sim) = e^{2\pi i x} \text{"="} (\cos 2\pi x, \sin 2\pi x) \in S$

$= \{x\}$ if $x \in (0, 1]$

$= \{0, 1\}$ if $x \in \{0, 1\}$

check

$e^{2\pi i 0} = e^{2\pi i \cdot 1}$

$(\cos 2\pi \cdot 0, \sin 2\pi \cdot 0) = (\cos 2\pi \cdot 1, \sin 2\pi \cdot 1)$

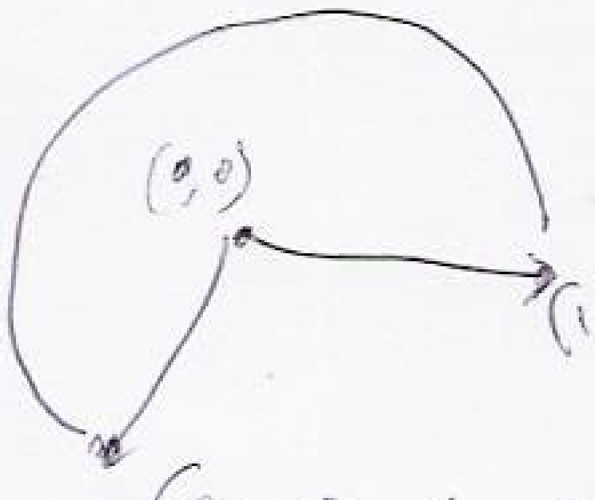
$(\cos 0, \sin 0) = (\cos 2\pi, \sin 2\pi)$

For every $(x, y) \in S$,

there is a unique $\theta \in [0, 2\pi)$ such that

$(\cos\theta, \sin\theta) = (x, y)$

length θ



So, $f\left(\frac{\theta}{2\pi}/\sim\right) = (x, y)$

$\frac{\theta}{2\pi} \in [0, 1)$

f is surjective

f is injective

We just need to show f & f^{-1} are cts.

~~f is cts at every~~

Let $\mathcal{U}$ be an open subset of S. To show f is cts, we just need to show that $f^{-1}\mathcal{U}$ is cts.

$V = \{x \in [0,1] \mid e^{2\pi i x} \in \mathcal{U}\}$

$\downarrow$

is open in $[0,1]$ because $x \longmapsto e^{2\pi i x}$ is continuous.

(because is differentiable)

V open in $[0,1]$

$$[x \in V \ \& \ x \sim y] \Rightarrow \begin{cases} x = y \Rightarrow y \in V \\ \{x, y\} = \{0, 1\} \Rightarrow e^{2\pi i x} = e^{2\pi i y} = e^{2\pi i 0} = e^{2\pi i 1} = \end{cases}$$

$$\Rightarrow \text{“}1\text{” = “}(1,0)\text{” } \& \ e^{2\pi i x} \in \mathcal{U}] \Rightarrow e^{2\pi i y} \in \mathcal{U} \Rightarrow y \in V$$

Similarly $[x \notin V \ \& \ x \sim y] \Rightarrow y \notin V$.

Therefore, V is open and a union of equivalence classes.

So, $f^{-1}\mathcal{U} = \{x/\sim \mid e^{2\pi i x} \in \mathcal{U}\} = \underbrace{\{x/\sim \mid x \in V\}}_{V/\sim}$ is open in $T = [0,1]/\sim$

Theorem 26.6

Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

f & f^{-1} cts. and bijective

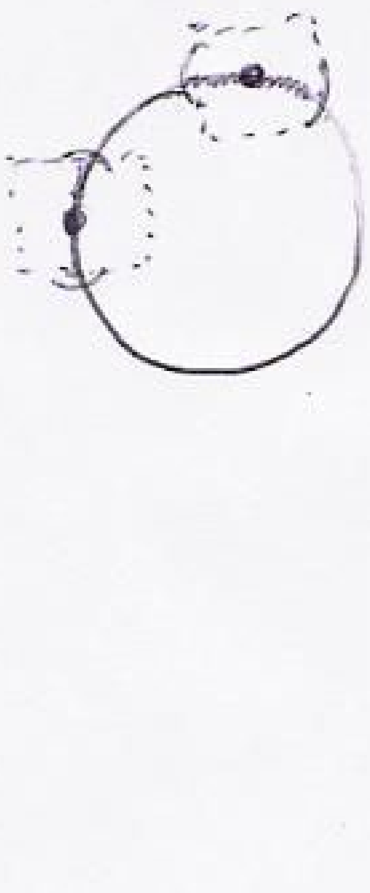
So enough to show that T is compact & S is Hausdorff.

The quotient map $x \mapsto x/\sim$ from $[0,1]$ to T is cts. (p. 137)

Every quotient ~~map~~ is cts. and surjective!

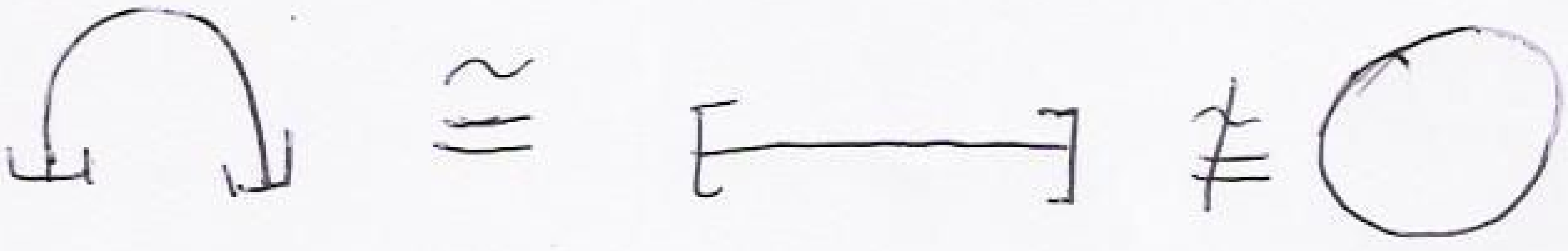
So, T is a ~~cts.~~ image of the compact space $[0,1]$.

So, T is compact too (26.5)



S

$\mathbb{R}^2$ is Hausdorff, and S is a ~~$~~subspace of $\mathbb{R}^2$, so S is Hausdorff (17.11)

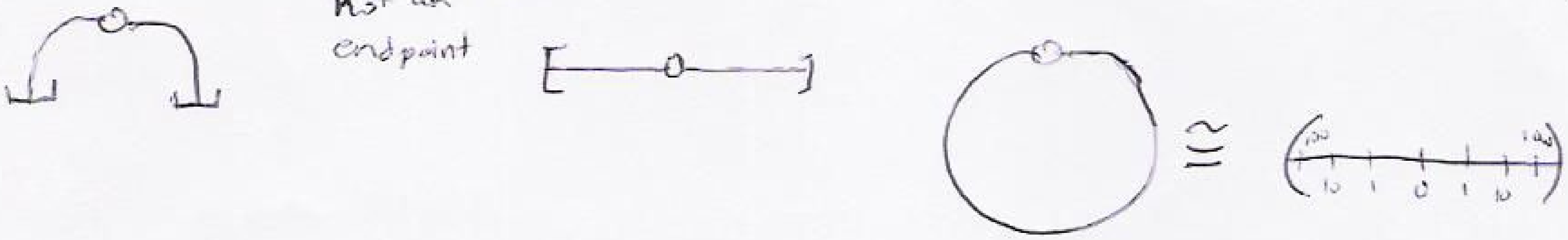


$H \cong I \ncong C$

H I C

$H - \{p\}$ $\quad I - \{p\}$ $\quad C - \{p\}$

not an endpoint



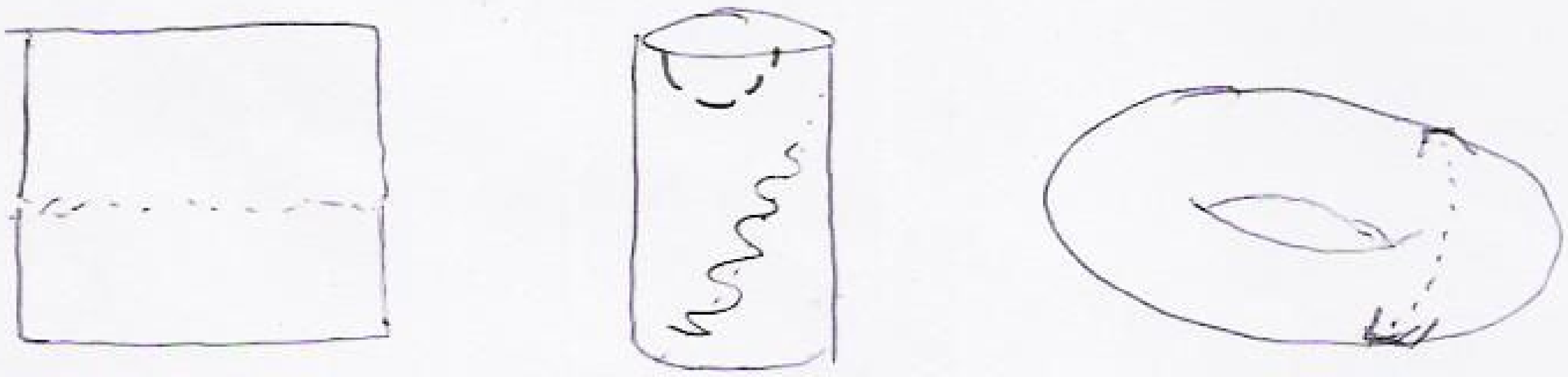
$\cong \mathbb{R}$

$\frac{x^2}{1+x^2} \longmapsfrom x$

$(-1,1) \cong \mathbb{R}$

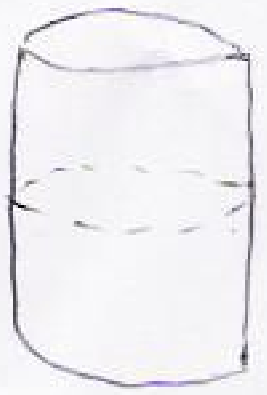
$[0,1]^2$ $\quad C \times [0,1]$ $\quad C^2$

Remove a closed interval

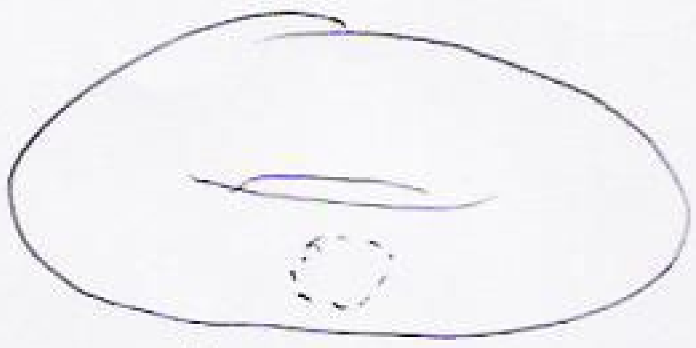


disconnected disconnected connected

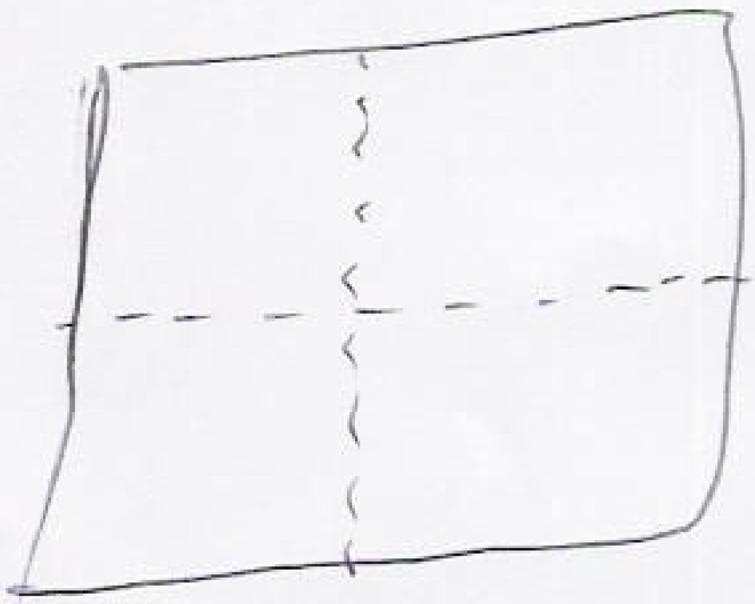
ove a circle



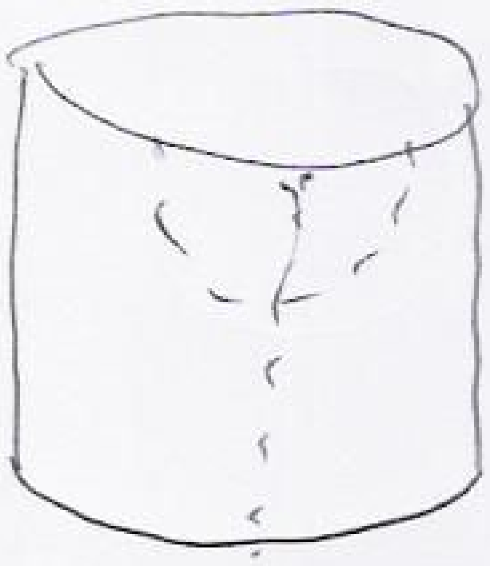
disconnected



disconnected



4 pieces
is possible



≤ 3 pieces