HW9

1. Let X be a space with topology generated by metric d, $p \in X$, and $\emptyset \neq A \subset X$. Prove that $p \in \bar{A} \iff d(p, A) = 0$ where $d(p, A) = \inf\{d(p, a) \mid a \in A\}$.
(inf = "greatest lower bound of")

2. Prove that if X & Y are metrizable spaces, then $X \times Y$ (with the product topology) is metrizable.
Hint: analog of uniform metric on $\mathbb{R}^2$.

3. Let $I = [0, 1]$ (with the subspace topology from $\mathbb{R}$),
$Y = I^I = \{f \mid f : I \to I\}$, and $X = \{f \in Y \mid f \text{ continuous}\}$.
$\mathcal{T}_1$ = topology on X generated by metric (called the L^1 metric)
$$d_1(f, g) = \int_0^1 |f(t) - g(t)| \, dt.$$
$\mathcal{T}_p$ = subspace topology on X from product topology on Y.

$\mathcal{T}_\infty$ = topology on X generated by $d_\infty(f, g) = \sup\{|f(t) - g(t)| \mid t \in I\}$.

HW9

Give an example of $f_1, f_2, f_3, \ldots \in X$ and $g_1, g_2, g_3, \ldots \in X$ such that $f_1, f_2, f_3, \ldots$ converges to the zero function $(0(x)=0)$ in $\mathcal{T}_1$ but not in $\mathcal{T}_p$ and $g_1, g_2, g_3, \ldots$ converges to 0 in $\mathcal{T}_p$ but not in $\mathcal{T}_\infty$.

Lemma (helpful for 3). Let $X = \prod_{\alpha \in J} X_\alpha$ have a product topology and $f_1, f_2, f_3, \ldots \in X$ and $g \in X$. Then $(f_n)_{n\in \mathbb{N}}$ converges to g if and only if, for all $\alpha \in J$, $(f_n(\alpha))_{n \in \mathbb{N}}$ converges to $g(\alpha)$.

Proof. "only if": Assuming $(f_n)_{n\in\mathbb{N}}$ converges to g, $\alpha \in J$, and U a nhbd. of $g(\alpha)$, $V = \pi_\alpha^{-1}(U)$ is a nhbd of g; hence, some tail $f_N, f_{N+1}, f_{N+2}, \ldots$ is in V; hence, $f_N(\alpha), f_{N+1}(\alpha), f_{N+2}(\alpha), \ldots$ is in U.

"if": Assuming each $(f_n(\alpha))_{n\in\mathbb{N}}$ converges to $g(\alpha)$, and $W = \pi_{\alpha_1}^{-1}(U_1) \cap \cdots \cap \pi_{\alpha_m}^{-1}(U_m)$ is a basic nhbd. of g, each U_i contains a tail $(f_n(\alpha_i))_{n \geq N_i}$ because U_i is a nhbd. of $g(\alpha_i)$; hence, W contains the tail $(f_n)_{n \geq N_0}$ where $N_0 = \max\{N_1, \ldots, N_m\}$. ∎