HW 17

1. Prove that if X is any space (even one that is not T_0) then there is a continuous bijection $f: Y \to X$ from a metrizable (hence, T_6) space Y. (Thus, continuous images don't preserve any separation axioms.)

2. Prove that if (X,d) is a metric space and $\emptyset \neq A \subset X$, then $d_A : X \to \mathbb{R}$ defined by $d_A(p) = \inf\{d(a,p) \mid a \in A\}$ is continuous.

3. Assuming the result of (2), prove that every metrizable space X is <u>perfectly normal</u>, that is, if $A \subset X$ is closed, then $A = f^{-1}(\{0\})$ for some continuous $f: X \to \mathbb{R}$.

4. Prove that perfect normality implies hereditary normality (if $A \cap \overline{B} = \emptyset = \overline{A} \cap B$, then A & B have disjoint open supersets ($U \supset A$ & $V \supset B$ & $U \cap V = \emptyset$)).

5 [Grad only] Prove that $\overline{S_\Omega} (= S_\Omega \cup \{\Omega\})$ is not perfectly normal.

Comment: 5 above implies that $\overline{S_\Omega}$ is T_5 but not T_6, implying T_5 & T_6 are different properties. ($\overline{S_\Omega}$ is T_5 because its topology is an order topology.)

Hint for 5: Prove that if $f : \overline{S_\Omega} \to \mathbb{R}$ is continuous, then f is constant on $[\alpha, \Omega]$ for some $\alpha < \Omega$. This will imply that $\{\Omega\}$, though closed, is not $f^{-1}(\{0\})$. (Do you know why $\{\Omega\}$ is closed? Prove that too.)