This homework is Grad. only.

HW 21

Let $X = C(\mathbb{R}, \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$.

Call $S \subset X$ <u>linearly independent</u> if for all $(a_1, \dots, a_n) \in \mathbb{R}^{<\omega}$ and all $f_1, \dots, f_n \in S$, if $(a_1, \dots, a_n) \neq (0, \dots, 0)$, then $a_1 f_1(x) + \dots + a_n f_n(x)$ is not 0 for some $x \in \mathbb{R}$.

1. Use Zorn's Lemma to prove that there is a maximal linearly independent $M \subset X$.

2. Given: M as in 1 and $g \in X$. Prove that there exist $(b_1, \dots, b_n) \in \mathbb{R}^{<\omega}$ and $f_1, \dots, f_n \in M$ such that $\forall x \in \mathbb{R}$ $g(x) = b_1 f_1(x) + \dots + b_n f_n(x)$.

(If you succeed, you've proven X has a (vector space) basis.)