BRANCH PRODUCT TOPOLOGIES

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Abstract. We define branch product topologies, a new product construction. Branch product topologies generalize sum topologies and product topologies. We investigate criteria for branch products or iterations of branch products to preserve various topological properties, focusing on the separation axioms and compactness. In particular, we generalize the Tychonoff Theorems for compactness and $\mathcal{D}$-compactness.

Introduction

Product topologies may be defined by declaring open the preimages of open sets by coordinates projection maps. We might visualize such a preimage as the set of strings passing through a hole of "open" shape. There is a similar way to topologize the set of maximal chains of a poset. We choose as special certain subsets of the poset. Then, for each special subset, we declare open the set of maximal chains intersecting that subset. This construction, which we call the branch product topology, generalizes both the product topology and sum topology. Going further, we may perform a kind of of iteration of the branch product construction which also generalizes sums and products.

Naturally, we investigate whether various classes of topologies preserved by products and sums are also preserved by branch products and iterations thereof. Our results for branch products are mixed. For iterations of branch products, which are actually equivalent to branch products of a special form, all our results are positive except for a specifically constructed counterexample.

Since many important topological classes are not preserved by both products and sums, we look for restrictions on branch products and iterations thereof that allow for nontrivial extensions of known preservation theorems for these properties. To get positive results about topological properties not preserved by products, such as normality, we examine the special case of branch product topologies of maximal chains of trees, which still generalize sum topologies. To get positive results about compactness, which is not preserved by sums, we examine restricted forms of branch product iterations that generalize products and finite sums but not arbitrary sums.

1. Preliminaries

Definition 1.1. Let X be a nonempty poset. A *branch* of X is a maximal chain of X. Let $\mathscr{B}(X)$ denote the set of all branches of X. A *semibranch* is an initial segment of a branch. A semibranch is *proper* if it is not a branch. Let $\mathscr{S}(X)$ denote the set of all proper semibranches of X. For each $S \in \mathscr{S}(X)$, define the *fork* of S to be the set of minimal strict upper bounds of S, and denote the fork of S by

Date: August 24, 2004.

$\mathscr{F}_X(S)$. For each $E \subseteq X$, let $\mathscr{B}_X(E)$ denote the set of branches of X that intersect E.

Remark 1.2. From the above definitions it is immediate that the every fork is an antichain; hence, the intersection of a fork and a chain contains at most one element.

Definition 1.3. By the previous remark, if C is a chain in X and $S \in \mathscr{S}(X)$ and $C \cap \mathscr{F}_X(S) \neq \emptyset$, then $C \cap \mathscr{F}_X(S)$ is a singleton, and we denote its element by $C@S$. For each $S \in \mathscr{S}(X)$, let π_S denote the map from $\mathscr{B}_X(\mathscr{F}_X(S))$ to $\mathscr{F}_X(S)$ given by $\pi_S(A) = A@S$ for all $A \in \mathscr{B}_X(\mathscr{F}_X(S))$.

We henceforth assume X is well-founded. Thus, $\mathscr{F}_X(S) \neq \emptyset$ for all $S \in \mathscr{S}(X)$. Moreover, all semibranches are well-ordered, allowing us to make the following definition.

Definition 1.4. For every $\alpha \in \mathrm{O}n$, we define S_α to be the unique semibranch R such that $R \subseteq S$ and the order type of R is the minimum of the order type of S and the order type of α. Define $h(S)$ to be the minimum ordinal α for which $S_\alpha = S$.

From this definition, we can immediately conclude the following proposition. The proof is a simple application of transfinite induction.

Proposition 1.5. *Let S be a semibranch and let $\alpha \in \mathrm{O}n$.*

(1) *If $\alpha < h(S)$, then $S_{\alpha+1} = S \cup \{S@S_\alpha\}$.*
(2) *If α is a limit ordinal, then $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$.*
(3) *A branch B can be defined recursively by defining $B@B_\alpha$ in terms of B_α for each $\alpha < h(B)$. More precisely, given a function $f\colon \mathscr{S}(X) \to X$ such that $f(S) \in \mathscr{F}_X(S)$ for all $S \in \mathscr{S}(X)$, there is a unique branch B such that $B@B_\alpha = f(B_\alpha)$ for all $\alpha < h(B)$.*

Branch product topologies over X are constructed by giving each fork of X a topology and then using these topologies to a topologies on the set of branches $\mathscr{B}(X)$. As a subset of X, a fork is simply referred to as a fork, but in the context of its topology, it is referred to as a *fork space*. The formal definition of branch product topologies is given below.

Definition 1.6. For each $S \in \mathscr{S}(X)$, let $\mathcal{T}_S$ be a topology on $\mathscr{F}_X(S)$. Thus, $\langle \mathscr{F}_X(S), \mathcal{T}_S \rangle$ is a fork space. We define the *branch product topology* on $\mathscr{B}(X)$ by declaring $\{\mathscr{B}_X(U) : S \in \mathscr{S}(X),\ U \in \mathcal{T}_S\}$ to be a subbasis of open sets. When the topologies of the fork spaces are not clear from the context, we use $\mathscr{B}\big(X, \langle \mathcal{T}_S \rangle_{S\in\mathscr{S}(X)}\big)$ to denote the set $\mathscr{B}(X)$ together with the branch product topology induced by the topologies $\langle \mathcal{T}_S \rangle_{S\in\mathscr{S}(X)}$.

As shown by the next two examples, branch products generalize sums and products.

Example 1.7. Let $\alpha \in \mathrm{O}n$ and let X_β be a nonempty topological space for each $\beta < \alpha$. Assume that these spaces are pairwise disjoint. Let $X = \bigcup_{\beta<\alpha} X_\beta$. For all $p, q \in X$, we declare $p \leq q$ if $p = q$ or there exist $\beta, \gamma \in \mathrm{O}n$ such that $\gamma < \beta < \alpha$ and $p \in X_\gamma$ and $q \in X_\beta$. Then every fork is equal to X_β for some $\beta < \alpha$, and we give this fork the corresponding topology. Then the branch product topology makes $\mathscr{B}(X)$ homeomorphic to $\prod_{\beta<\alpha} X_\beta$; hence, the branch product topology generalizes the product topology.

Example 1.8. Let A be a nonempty set. For each $a \in A$, let B_a be a nonempty set. Assume that these sets are pairwise disjoint. Let $X = A \cup \bigcup_{a\in A} B_a$. For all $p, q \in X$, we declare $p \leq q$ if $p = q$ or if $p \in A$ and $q \in B_p$. Then $\mathscr{S}(X) = \{\emptyset\} \cup \{\{a\} : a \in A\}$. If, for every a in A, we give B_a a topology and declare the $\mathcal{T}_{\{a\}}$ to be that topology, then, regardless of our choice for $\mathcal{T}_\emptyset$, the space $\mathscr{B}(X)$ is homeomorphic to $\bigcup_{a\in A} B_a$ with the sum topology. Hence, the branch product topology generalizes the sum topology.

The following theorem gives an equivalent formulation of the branch product topology in terms of continuous maps.

Theorem 1.9. *The branch product topology on $\mathscr{B}(X)$ is the coarsest topology on $\mathscr{B}(X)$ with the property that π_S is continuous and $\mathscr{B}_X(\mathscr{F}_X(S))$ is open for all $S \in \mathscr{S}(X)$.*

Proof. Let $S \in \mathscr{S}(X)$. Then π_S is clearly continuous with respect to the branch product topology on $\mathscr{B}(X)$. Indeed, if $U \in \mathcal{T}_S$, then $\pi_S^{-1}(U) = \mathscr{B}_X(U)$, which is open in the branch product topology. Also, $\mathscr{F}_X(S) \in \mathcal{T}_S$; hence, $\mathscr{B}_X(\mathscr{F}_X(S))$ is open in the branch product topology.

Conversely, suppose $\mathscr{O}$ is a topology on $\mathscr{B}(X)$ such that π_S is continuous and $\mathscr{B}_X(\mathscr{F}_X(S)) \in \mathscr{O}$ for all $S \in \mathscr{S}(X)$. Then, for each $S \in \mathscr{S}(X)$ and $U \in \mathcal{T}_S$, we have $\mathscr{B}_X(U) = \pi_S^{-1}(U)$; hence, there exists $\mathcal{U} \in \mathscr{O}$ such that $\mathcal{U} \cap \mathscr{B}_X(\mathscr{F}_X(S)) = \mathscr{B}_X(U)$. Since $\mathscr{B}_X(\mathscr{F}_X(S)) \in \mathscr{O}$, we have $\mathscr{B}_X(U) \in \mathscr{O}$. Therefore, $\mathscr{O}$ is finer than the branch product topology. □

As general as branch products are, they are clearly ways to generalize them. Instead of using the open subsets of fork spaces of X to induce a topology on $\mathscr{B}(X)$, why not use other subsets of X? Furthermore, if we do not require these other subsets to have anything to do with forks, then why require X to well-founded? The following definition makes these generalizations. Before we state it, let us agree that Y henceforth denotes a nonempty and not necessarily well-founded poset.

Definition 1.10. Given $\mathcal{E} \subseteq \mathcal{P}(Y)$ such that $\mathscr{B}_Y(\bigcup \mathcal{E}) = \mathscr{B}(Y)$, let $\mathscr{B}(Y, \mathcal{E})$ denote $\mathscr{B}(Y)$ with the coarsest topology for which $\mathscr{B}_Y(E)$ is open for all $E \in \mathcal{E}$. The topology of $\mathscr{B}(Y, \mathcal{E})$ is called the $\mathcal{E}$-*induced* branch product topology.

The requirement $\mathscr{B}_Y(\bigcup \mathcal{E}) = \mathscr{B}(Y)$ ensures that $\{\mathscr{B}_Y(E) : E \in \mathcal{E}\}$ is a subbasis of open sets for $\mathscr{B}(Y, \mathcal{E})$. Moreover, using this notation, we have

$$\mathscr{B}\left(X, \bigcup_{S\in\mathscr{S}(X)} \mathcal{T}_S\right) = \mathscr{B}\big(X, \langle \mathcal{T}_S \rangle_{S\in\mathscr{S}(X)}\big)\,.$$

Thus, the branch product topology of Definition 1.6 is just a particular $\mathcal{E}$-*induced* branch product topology; we might call it the fork-space-induced branch product topology.

The drawback of generalizing to arbitrary $\mathcal{E}$-induced branch products is that it doesn't make sense to say whether they preserve a topological property, for $\mathcal{E}$ is generally just a subset of $\mathcal{P}(Y)$, not a collection of topological spaces such as fork spaces. However, it still makes sense to say whether a branch product iteration (defined in Section 4) preserves a topological property in the more general setting of $\mathcal{E}$-induced branch products. Moreover, all of our results about branch product iterations preserving topological properties hold in this more general setting.

Of course, they are many other ways to topologize set of branches. For example, given $\mathcal{E} \subset \mathcal{P}(Y)$, consider $\mathscr{B}(Y)$ the weakest topology for which $\mathscr{B}_Y(E)$ is closed for all $E \in \mathcal{E}$. Let $\mathscr{B}^c(Y, \mathcal{E})$ denote $\mathscr{B}(Y)$ with this topology, which we call the *$\mathcal{E}$-induced closed branch product topology.* Then $\mathscr{B}^c(X)$ is homeomorphic to $\prod_{\beta<\alpha} X_\beta$ in Example 1.7. Moreover, although we will not prove this in this paper, the Tychonoff Theorem has generalizations both for branch product iterations and for closed branch product iterations. But $\mathscr{B}^c(X)$ is not generally homeomorphic to $\bigcup_{a\in A} B_a$ with the sum topology in Example 1.8. Moreover, while branch product topologies are shown to preserve the T_2 axiom in Section 2, we also there give an example of an X for which $\mathscr{B}^c(X)$ is not Hausdorff, even if all the fork spaces of X are discrete. For these reasons, we primarily investigate the branch product topology.

Let us briefly mention one other topologization of $\mathscr{B}(X)$. Consider sets of the form $\bigcap_{U\in\mathcal{E}} \mathscr{B}_X(U)$ where $\mathcal{E}$ is a set of disjoint open subsets of fork spaces of X. If the collection of all such sets is declared to be a subbasis, then $\mathscr{B}(X)$ remains homeomorphic to $\bigcup_{a\in A} B_a$ with the sum topology in Example 1.8, but becomes homeomorphic to $\prod_{\beta<\alpha} X_\beta$ with the box topology in Example 1.7. We might call this topologization of $\mathscr{B}(X)$ the branch box product topology.

2. Separation Axioms T_0 Through $T_{3\frac{1}{2}}$

In this section, we examine when branch products preserve separation axioms and when they break them.

The usual product topology preserves the separation axioms T_0 through $T_{3\frac{1}{2}}$. Sum topologies preserve these separation axioms as well. Which of the separation axioms are preserved by branch products, which generalize topological sums and cartesian products? In general, only T_0 through T_2 are preserved, as shown next.

Proposition 2.1. *Suppose $E, F \subseteq X$ and $E \cup F$ is an antichain. Then*

$$\mathscr{B}_X(E \cap F) = \mathscr{B}_X(E) \cap \mathscr{B}_X(F)\,.$$

In particular, if $S \in \mathscr{S}(X)$ and E and F are disjoint subsets of $\mathscr{F}_X(S)$, then $\mathscr{B}_X(E)$ and $\mathscr{B}_X(F)$ are disjoint.

Proof. That $\mathscr{B}_X(E\cap F) \subseteq \mathscr{B}_X(E) \cap \mathscr{B}_X(F)$ is clear. Let us show the reverse inclusion. Let $A \in \mathscr{B}_X(E) \cap \mathscr{B}_X(F)$. Then there exist $e \in A \cap E$ and $f \in A \cap F$. Since $E \cup F$ is an antichain and A is a chain, $e = f$. Thus, $A \in \mathscr{B}_X(E \cap F)$. □

Theorem 2.2. *Let $i \in \{0, 1, 2\}$. Let all the fork spaces of X satisfy the T_i axiom. Then $\mathscr{B}(X)$ satisfies the T_i axiom.*

Proof. The proof is a straightforward application of Proposition 2.1. Let A and B be distinct branches. Let S be the maximal semibranch contained in $A \cap B$. Then $A@S \neq B@S$. If $\mathscr{F}_X(S)$ is a T_0 space, then it has an open subset U containing exactly one of $A@S$ and $B@S$; hence, $\mathscr{B}_X(U)$ contains exactly one of A and B. If $\mathscr{F}_X(S)$ is a T_1 space, then it has an open subset U containing $A@S$ but not $B@S$; hence, $\mathscr{B}_X(U)$ contains A but not B. If $\mathscr{F}_X(S)$ is a T_2 space, then it has disjoint open subsets U and V such that $A@S \in U$ and $B@S \in V$; hence, $\mathscr{B}_X(U)$ and $\mathscr{B}_X(V)$ are disjoint, $A \in \mathscr{B}_X(U)$, and $B \in \mathscr{B}_X(V)$. □

In contrast to the above theorem, the next example shows that $\mathscr{B}^c(X)$ need not be Hausdorff, even when all fork spaces of X are discrete.

Example 2.3. Let $X = \bigcup_{n<\omega}\{a_n, b_n\} \cup \{c, d\}$. Define $\leq$ by requiring the following:

(1) $\mathscr{S}(X) = \{\{a_n : n < \alpha\} : \alpha \leq \omega\}$;
(2) $\mathscr{F}_X(\{a_n : n < \omega\}) = \{c, d\}$;
(3) $\mathscr{F}_X(\{a_m : m < n\}) = \{a_n, b_n\}$ for each $n < \omega$.

Give every fork of X the discrete topology. Then the collection of sets of the form $\mathscr{B}(X) \setminus \mathscr{B}_X(F)$ for a finite $F \subseteq X$ is a basis of open sets for $\mathscr{B}^c(X)$. Let C and D respectively denote the branches $\{a_n : n < \omega\} \cup \{c\}$ and $\{a_n : n < \omega\} \cup \{d\}$. Thus, if $\mathscr{B}^c(X)$ is Hausdorff, then there exists finite sets $E, F \subseteq X$ such that $C \notin \mathscr{B}_X(E)$ and $D \notin \mathscr{B}_X(F)$ and $\mathscr{B}_X(E \cup F) = \mathscr{B}(X)$. Suppose such E and F exist. Then $E \subseteq \{b_n : n < \omega\} \cup \{c\}$ and $F \subseteq \{b_n : n < \omega\} \cup \{d\}$. Moreover, since E and F are finite, there exists $k < \omega$ such that $b_k \notin E \cup F$. Hence,

$$\{a_j : j < k\} \cup \{b_k\} \notin \mathscr{B}_X(E \cup F),$$

which contradicts $\mathscr{B}_X(E \cup F) = \mathscr{B}(X)$. Thus, $\mathscr{B}^c(X)$ is not Hausdorff.

Let us return to considering $\mathscr{B}(X)$ with the branch product topology, and when it preserves the higher separation axioms.

Example 2.4. *Branch products do not preserve any of the separation axioms T_3 through T_6.* Let $\mathscr{F}_X(\emptyset) = \{0\} \times \mathbb{R}$ and give it the order topology of $\mathbb{R}$. Let $K = \{1/(n+1) : n < \omega\}$. For all $x \in \mathbb{R} \setminus K$, let $\mathscr{F}_X(\{\langle 0, x\rangle\}) = \{\langle 1, 0\rangle\}$. Let X have no other forks. Then $\mathscr{B}(X)$ is homeomorphic to $\mathbb{R}$ with the topology generated by the basis consisting of the open intervals and their intersections with $\mathbb{R} \setminus K$. In this topology, K is closed but 0 and K do not have disjoint open neighborhoods; hence, $\mathscr{B}(X)$ is not a T_3 space, despite the fact that all the fork spaces of X are T_6 (perfectly normal Hausdorff) spaces.

Working with an arbitrary nonempty tree instead of an arbitrary nonempty well-founded poset leads to many nice results. In particular, branch product topologies over trees preserve the T_3 and $T_{3\frac{1}{2}}$ axioms. Henceforth, let W be a nonempty tree.

Proposition 2.5. *The set $\{\mathscr{B}_W(U) : U \in \mathcal{T}_S, S \in \mathscr{S}(W)\}$ is a basis for $\mathscr{B}(W)$.*

Proof. It suffices to show that $\{\mathscr{B}_W(U) : U \in \mathcal{T}_S,\ S \in \mathscr{S}(W)\}$ is closed under pairwise intersections. Let $S, S' \in \mathscr{S}(W)$ and $U \in \mathcal{T}_S$ and $U' \in \mathcal{T}_{S'}$. Suppose there exists $B \in \mathscr{B}_W(U) \cap \mathscr{B}_W(U')$. Then $S, S' \subseteq B$. Thus, $S \subsetneq S'$ or $S' \subsetneq S$ or $S = S'$. In the first case

$$\mathscr{B}_W(U') \subseteq \mathscr{B}_W(\mathscr{F}_W(S')) \subseteq \mathscr{B}_W(\{S' @ S\}) \subseteq \mathscr{B}_W(U).$$

Whence, $\mathscr{B}_W(U) \cap \mathscr{B}_W(U') = \mathscr{B}_W(U')$. Likewise, $\mathscr{B}_W(U) \cap \mathscr{B}_W(U') = \mathscr{B}_W(U)$ in the second case. In the third case, $U \cup U' \subseteq \mathscr{F}_W(S)$; hence, $U \cup U'$ is antichain; hence, $\mathscr{B}_W(U) \cap \mathscr{B}_W(U') = \mathscr{B}_W(U \cap U')$ and $U \cap U' \in \mathcal{T}_S$. □

Lemma 2.6. *Suppose all fork spaces of W satisfy the T_1 axiom. Then, for all $S \in \mathscr{S}(W)$ and for all V closed in $\mathscr{F}_W(S)$, the subspace $\mathscr{B}_W(V)$ is closed.*

Proof. Suppose $S \in \mathscr{S}(W)$ and V is closed in $\mathscr{F}_W(S)$ and $A \in \mathscr{B}(W) \setminus \mathscr{B}_W(V)$. Let $T = A \cap S$. Suppose $T \subsetneq S$. Then $A@T \neq S@T$; hence, there exists $U \in \mathcal{T}_T$ such that $A@T \in U$ and $S@T \notin U$. Thus, $A \in \mathscr{B}_W(U)$ and, by Proposition 2.1, $\mathscr{B}_W(U)$ is disjoint from $\mathscr{B}_W(\{S@T\})$, which contains $\mathscr{B}_W(\mathscr{F}_W(S))$, which contains $\mathscr{B}_W(V)$. Suppose $S = T$. Then $A@S \notin V$; hence, there exists $U \in \mathcal{T}_S$ such that $A@S \in U$ and $U \cap V = \emptyset$. Thus, $A \in \mathscr{B}_W(U)$ and, by Proposition 2.1, $\mathscr{B}_W(U) \cap \mathscr{B}_W(V) = \emptyset$. □

Theorem 2.7. *Suppose all fork spaces of W satisfy the T_3 axiom. Then $\mathscr{B}(W)$ satisfies the T_3 axiom.*

Proof. Let $\mathcal{C}$ be a closed subset of $\mathscr{B}(W)$ and let A be a branch not in $\mathcal{C}$. By Proposition 2.5, there exist $S \in \mathscr{S}(W)$ and $U \in \mathcal{T}_S$ such that $A \in \mathscr{B}_W(U)$ and $\mathcal{C} \cap \mathscr{B}_W(U) = \emptyset$. Choose $V \in \mathcal{T}_S$ such that $A@S \in V$ and $\overline{V} \subseteq U$. By Lemma 2.6, $\mathscr{B}(W) \setminus \mathscr{B}_W\big(\overline{V}\big)$ is open. Moreover, $\mathscr{B}(W) \setminus \mathscr{B}_W\big(\overline{V}\big)$ contains $\mathcal{C}$ and is disjoint from $\mathscr{B}_W(V)$, which contains A. Thus, A and $\mathcal{C}$ are separated by open sets. □

Theorem 2.8. *Suppose all fork spaces of W satisfy the $T_{3\frac{1}{2}}$ axiom. Then $\mathscr{B}(W)$ satisfies the $T_{3\frac{1}{2}}$ axiom.*

Proof. Let $\mathcal{C}$ be a closed subset of $\mathscr{B}(W)$ and let A be a branch not in $\mathcal{C}$. By Proposition 2.5, there exist $S \in \mathscr{S}(W)$ and $U \in \mathcal{T}_S$ such that $A \in \mathscr{B}_W(U)$ and $\mathcal{C} \cap \mathscr{B}_W(U) = \emptyset$. Set $V = \mathscr{F}_W(S) \cap \bigcup \mathcal{C}$. Then $U \cap V = \emptyset$; hence, $U \cap \overline{V} = \emptyset$. Choose $f\colon \mathscr{F}_W(S) \to [0,1]$ such that f is continuous, $f(A@S) = 1$, and $f(\overline{V}) \subseteq \{0\}$. Define $g\colon \mathscr{B}_W(\mathscr{F}_W(S)) \to [0,1]$ by $g = f \circ \pi_S$. Then g is continuous. By Lemma 2.6, $\mathscr{B}_W(\mathscr{F}_W(S))$ is clopen; hence, we may extend g to a continuous map from $\mathscr{B}(W)$ to $[0,1]$ by setting $g(B) = 0$ for all $B \in \mathscr{B}(W) \setminus \mathscr{B}_W(\mathscr{F}_W(S))$. Then $g(A) = 1$ and $g(\mathcal{C}) \subseteq \{0\}$. □

3. Nonjaggedness

Branch product topologies over trees do not preserve the T_4 axiom. Indeed, we have the following general result.

Theorem 3.1. *Suppose $\mathcal{M}$ is a class of topological spaces and $\mathcal{M}$ is hereditary with respect to closed subspaces but not with respect to all subspaces. Then there is a tree W with all its fork spaces are in $\mathcal{M}$, but $\mathscr{B}(W)$ is not in $\mathcal{M}$.*

Proof. First, note that $\mathcal{M}$ must contain a nonempty topological space, for all subspaces of the empty space are closed. Therefore, we may choose $\mathscr{F}_W(\emptyset)$ to be a topological space in $\mathcal{M}$ with a subspace K that is not in $\mathcal{M}$. For each $p \in \mathscr{F}_W(\emptyset) \setminus K$, let $\mathscr{F}_W(\{p\})$ be an arbitrary nonempty topological space in $\mathcal{M}$. Let W have no other forks. Then $\mathscr{B}_W(\{p\}) = \{\{p\}\}$ for all $p \in K$; hence, $\mathscr{B}_W(K)$ and K are homeomorphic when given their respective subspace topologies; hence, $\mathscr{B}_W(K) \notin \mathcal{M}$. Since $\mathscr{B}_W(K) = \mathscr{B}(W) \setminus \bigcup_{p \in \mathscr{F}_W(\emptyset) \setminus K} \mathscr{B}_W(\mathscr{F}_W(\{p\}))$, the subspace $\mathscr{B}_W(K)$ is closed in $\mathscr{B}(W)$. Therefore, $\mathscr{B}(W) \notin \mathcal{M}$. □

Many topological properties are hereditary with respect to closed subspaces but not with respect to all subspaces. Examples include normality, compactness, local compactness, paracompactness, and most variants of compactness and paracompactness. Thus, the negative result of Theorem 3.1 applies to all these properties. The crucial part of the proof of this theorem is the construction of a pathological closed subspace. By making a mild additional assumption, we can often avoid such subspaces.

Definition 3.2. We say that a fork space of W is *nonjagged* if the subspace of points in the fork that are maximal in W is closed.

Theorem 3.3. *Suppose all fork spaces of W are normal and nonjagged. Then $\mathscr{B}(W)$ is normal.*

Proof. Let $\mathcal{V}$ and $\mathcal{V}'$ be disjoint closed subsets of $\mathscr{B}(W)$. For each $S \in \mathscr{S}(W)$, set

$$V_S = \{p \in \mathscr{F}_W(S) : \mathcal{V} \cap \mathscr{B}_W(\{p\}) \neq \emptyset\},$$

and define V'_S analogously. Let M_S denote the set of elements of V_S that are maximal in W; define M'_S analogously.

Let us show that $\mathscr{B}_W\big(\overline{M_S}\big) \subseteq \mathcal{V}$. Suppose $p \in \overline{M_S}$. Then p is maximal in W; hence, there is a unique branch A that contains p. Suppose $A \notin \mathcal{V}$. Then there exist $T \in \mathscr{S}(W)$ and $U \in \mathcal{T}_T$ such that $A \in \mathscr{B}_W(U)$ and $\mathcal{V} \cap \mathscr{B}_W(U) = \emptyset$. In particular, $S \cup \{q\} \notin \mathscr{B}_W(U)$ for all $q \in M_S$. If $T = S$, then $U \in \mathcal{T}_S$; whence, $U \cap M_S = \emptyset$; whence, $U \cap \overline{M_S} = \emptyset$; whence, $p \notin U$; whence, $A \notin \mathscr{B}_W(U)$, which is absurd. Therefore, $T \neq S$. Moreover, since p is the maximum of A, we have $T \subseteq S$. Thus, $T \subsetneq S$. Thus, for all $q \in M_S$, we have

$$S \cup \{q\} \in \mathscr{B}_W(\mathscr{F}_W(S)) \subseteq \mathscr{B}_W(\{S@T\}) = \mathscr{B}_W(\{A@T\}) \subseteq \mathscr{B}_W(U)\,,$$

which is absurd. Therefore, $\mathscr{B}_W\big(\overline{M_S}\big) \subseteq \mathcal{V}$. Similarly, $\mathscr{B}_W\big(\overline{M'_S}\big) \subseteq \mathcal{V}'$.

Let us show that $\overline{M_S} \cap \overline{V'_S} = \emptyset$. Suppose $p \in \overline{M_S} \cap \overline{V'_S}$. Set $A = S \cup \{p\}$. Since $p \in \overline{M_S}$, the point p is maximal in W. Therefore, $A \in \mathscr{B}_W\big(\overline{M_S}\big) \subseteq \mathcal{V}$. Hence, $A \notin \mathcal{V}'$. Hence, there exist $T \in \mathscr{S}(W)$ and $U \in \mathcal{T}_T$ such that $A \in \mathscr{B}_W(U)$ and $\mathscr{B}_W(U) \cap \mathcal{V}' = \emptyset$. Clearly, $T \subseteq S$. If $T = S$, then $U \cap V'_S = \emptyset$; whence, $p \notin \overline{V'_S}$, which is absurd. Thus, $T \subsetneq S$. Since $p \in \overline{V'_S}$, we have $V'_S \neq \emptyset$. Therefore, choose $q \in V'_S$ and $B \in \mathscr{B}_W(\{q\}) \cap \mathcal{V}'$. Then

$$B \in \mathscr{B}_W(\mathscr{F}_W(S)) \subseteq \mathscr{B}_W(\{S@T\}) = \mathscr{B}_W(\{A@T\}) \subseteq \mathscr{B}_W(U)\,.$$

Hence, $\mathscr{B}_W(U) \cap \mathcal{V}' \neq \emptyset$, a contradiction. Therefore, $\overline{M_S} \cap \overline{V'_S} = \emptyset$. Similarly, $\overline{M'_S} \cap \overline{V_S} = \emptyset$.

Choose $Y'_S, Z_S \in \mathcal{T}_S$ such that $\overline{V'_S} \subseteq Y'_S$ and $\overline{M_S} \subseteq Z_S$ and $Y'_S \cap Z_S = \emptyset$. Similarly, choose $Y_S, Z'_S \in \mathcal{T}_S$ such that $\overline{V_S} \subseteq Y_S$ and $\overline{M'_S} \subseteq Z'_S$ and $Y_S \cap Z'_S = \emptyset$. Set $U_S = Y_S \cap Z_S$ and $U'_S = Y'_S \cap Z'_S$. Then $U_S \cap U'_S = \emptyset$. Moreover, since $M_S \subseteq V_S$, we have $\overline{M_S} \subseteq Y_S$; hence, $\overline{M_S} \subseteq U_S$. Similarly, $\overline{M'_S} \subseteq U'_S$. Moreover, since $V'_S \subseteq Y'_S$ and $U_S \cap Y'_S \subseteq Z_S \cap Y'_S = \emptyset$, we have $U_S \cap V'_S = \emptyset$. Similarly, $U'_S \cap V_S = \emptyset$.

Define $\mathcal{A}$ to be the set of $S \in \mathscr{S}(W)$ for which $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathcal{V} \neq \emptyset$ and $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathcal{V}' = \emptyset$. Similarly, define $\mathcal{A}'$ to be the set of $S \in \mathscr{S}(W)$ for which $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathcal{V}' \neq \emptyset$ and $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathcal{V} = \emptyset$. Also, define $\mathcal{A}''$ to be the set of $S \in \mathscr{S}(W)$ for which $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathcal{V} \neq \emptyset$ and $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathcal{V}' \neq \emptyset$. Set

$$\mathcal{U} = \bigcup_{S \in \mathcal{A}} \mathscr{B}_W(\mathscr{F}_W(S)) \cup \bigcup_{S \in \mathcal{A}''} \mathscr{B}_W(U_S) \text{ and}$$

$$\mathcal{U}' = \bigcup_{S \in \mathcal{A}'} \mathscr{B}_W(\mathscr{F}_W(S)) \cup \bigcup_{S \in \mathcal{A}''} \mathscr{B}_W(U'_S)\,.$$

Then $\mathcal{U}$ and $\mathcal{U}'$ are open. Let us show they are disjoint. For each $S \in \mathcal{A}$ and $S' \in \mathcal{A}'$, we have $\mathscr{B}_W(\mathscr{F}_W(S)) \nsubseteq \mathscr{B}_W(\mathscr{F}_W(S')) \nsubseteq \mathscr{B}_W(\mathscr{F}_W(S))$. Since W is a tree, we have $S \nsubseteq S' \nsubseteq S$. Therefore, again using the fact that W is a tree, we have $\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathscr{B}_W(\mathscr{F}_W(S')) = \emptyset$.

Suppose $S \in \mathcal{A}''$ and $S' \in \mathcal{A}'$. Then $S \neq S'$. If $S \nsubseteq S' \nsubseteq S$, then

$$\mathscr{B}_W(\mathscr{F}_W(S)) \cap \mathscr{B}_W(\mathscr{F}_W(S')) = \emptyset;$$

Whence, $\mathscr{B}_W(U_S)\cap\mathscr{B}_W(\mathscr{F}_W(S'))=\emptyset$. Suppose $S'\subsetneq S$. Let $A\in\mathscr{B}_W(\mathscr{F}_W(S))\cap\mathcal{V}$. Since $\mathscr{B}_W(\mathscr{F}_W(S))\subseteq\mathscr{B}_W(\{S@S'\})=\mathscr{B}_W(\{A@S'\})$, we have $A\in\mathscr{B}_W(\mathscr{F}_W(S'))$, in contradiction with the definition of $\mathcal{A}'$. Therefore, $S\subsetneq S'$. Consequently, $\mathscr{B}_W(\mathscr{F}_W(S'))\subseteq\mathscr{B}_W(\{S'@S\})$. Hence, by the definitions of $\mathcal{A}'$ and V'_S, we have $S'@S\in V'_S$; hence, $S'@S\notin U_S$. By Proposition 2.1, $\mathscr{B}_W(U_S)\cap\mathscr{B}_W(\{S'@S\})=\emptyset$. Thus, $\mathscr{B}_W(U_S)\cap\mathscr{B}_W(\mathscr{F}_W(S'))=\emptyset$. Similarly, if $S\in\mathcal{A}$ and $S'\in\mathcal{A}''$, then $\mathscr{B}_W(\mathscr{F}_W(S))\cap\mathscr{B}_W(U'_{S'})=\emptyset$.

Now suppose $S,S'\in\mathcal{A}''$. If $S\not\subseteq S'\not\subseteq S$, then $\mathscr{B}_W(\mathscr{F}_W(S))\cap\mathscr{B}_W(\mathscr{F}_W(S'))=\emptyset$; whence, $\mathscr{B}_W(U_S)\cap\mathscr{B}_W(U'_{S'})=\emptyset$. If $S=S'$, then $U_S\cap U'_{S'}=\emptyset$; whence, by Proposition 2.1, $\mathscr{B}_W(U_S)\cap\mathscr{B}_W(U'_{S'})=\emptyset$. If $S\subsetneq S'$, then $\mathscr{B}_W(U'_{S'})\subseteq\mathscr{B}_W(\{S'@S\})$; whence, $S'@S\in V'_S$; whence, $S'@S\notin U_S$; whence, by Proposition 2.1,

$$\mathscr{B}_W(U_S)\cap\mathscr{B}_W(U'_{S'})\subseteq\mathscr{B}_W(U_S)\cap\mathscr{B}_W(\{S'@S\})=\emptyset.$$

Similarly, if $S'\subsetneq S$, then $\mathscr{B}_W(U_S)\cap\mathscr{B}_W(U'_{S'})=\emptyset$. Therefore, $\mathcal{U}\cap\mathcal{U}'=\emptyset$.

Let us show that $\mathcal{V}\subseteq\mathcal{U}$. Suppose $A\in\mathcal{V}$. Since $A\notin\mathcal{V}'$, there exist $S\in\mathscr{S}(W)$ and $U\in\mathcal{T}_S$ such that $A\in\mathscr{B}_W(U)$ and $\mathscr{B}_W(U)\cap\mathcal{V}'=\emptyset$. Suppose $A@S$ is not maximal in W. Then $\mathscr{B}_W(\mathscr{F}_W(S\cup\{A@S\}))=\mathscr{B}_W(\{A@S\})\subseteq\mathscr{B}_W(U)$. Hence, $\mathscr{B}_W(\mathscr{F}_W(S\cup\{A@S\}))\cap\mathcal{V}'=\emptyset$; hence, $S\cup\{A@S\}\in\mathcal{A}$; hence, $A\in\mathcal{U}$. Suppose $A@S$ is maximal in W. Then $A@S\in M_S$; hence, $A@S\in U_S$; hence, $A\in\mathcal{U}$. Thus, $\mathcal{V}\subseteq\mathcal{U}$. Similarly, $\mathcal{V}'\subseteq\mathcal{U}'$. Therefore, $\mathcal{V}$ and $\mathcal{V}'$ are separated by open sets. □

Theorem 3.4. *Suppose all fork spaces of W are paracompact and nonjagged. Then $\mathscr{B}(W)$ is paracompact.*

Proof. Let $\mathbb{U}$ be an open cover of $\mathscr{B}(W)$. We inductively define a locally finite open refinement of $\mathbb{U}$ with union $\mathscr{B}(W)$. The existence of such a refinement suffices to prove the theorem.

Suppose $\alpha\in\mathrm{O}n$ and we have the following:

(1) $\{\mathcal{A}_\beta:\beta<\alpha\}$ is a pairwise disjoint set of $\subseteq$-antichains of proper semibranches;
(2) if $\beta<\alpha$ and $S\in\mathcal{A}_\beta$ and T is a proper semibranch contained in S, then there exists $\gamma\leq\beta$ such that $T\in\mathcal{A}_\gamma$;
(3) if $\beta<\alpha$ and $S\in\mathcal{A}_\beta$, then $\mathcal{E}_{S,\beta}$ is a subset of $\mathcal{T}_S$ whose union contains all elements of $\mathscr{F}_W(S)$ that are maximal in W;
(4) $\{\mathscr{B}_W(U):U\in\mathcal{E}_{S,\beta}\}$ is a locally finite refinement of $\mathbb{U}$ for each $\beta<\alpha$ and $S\in\mathcal{A}_\beta$;
(5) if $\beta,\gamma<\alpha$ and $S\in\mathcal{A}_\beta$ and $T\in\mathcal{A}_\gamma$ and $T\subsetneq S$, then $S@T\notin\bigcup\mathcal{E}_{T,\gamma}$.

Let $\mathcal{B}_\alpha$ be the set of $\subseteq$-minimal elements of $\mathscr{S}(W)\setminus\bigcup_{\beta<\alpha}\mathcal{A}_\beta$. Let $\mathcal{A}_\alpha$ be the set of $S\in\mathcal{B}_\alpha$ for which there is no $\beta<\alpha$ and $T\in\mathcal{A}_\beta$ such that $T\subsetneq S$ and $S@T\in\bigcup\mathcal{E}_{T,\beta}$. Suppose $S\in\mathcal{A}_\alpha$. If there exist $T\in\mathscr{S}(W)$ and $\mathcal{U}\in\mathbb{U}$ such that $T\subsetneq S$ and $\mathscr{B}_W(\{S@T\})\subseteq\mathcal{U}$, then set $\mathcal{E}_{S,\alpha}=\{\mathscr{F}_W(S)\}$. Suppose such T and $\mathcal{U}$ do not exist. Let M_S denote the set of elements of $\mathscr{F}_W(S)$ that are maximal in W. If $M_S=\emptyset$, then set $\mathcal{E}_{S,\alpha}=\emptyset$. Suppose $M_S\neq\emptyset$. Then, for each $p\in M_S$, there exist $U_p\in\mathcal{T}_S$ and $\mathcal{U}_p\in\mathbb{U}$ such that $p\in U_p$ and $\mathscr{B}_W(U_p)\subseteq\mathcal{U}_p$. Since M_S is closed and $\mathscr{F}_W(S)$ is paracompact, we may choose $\mathcal{E}_{S,\alpha}$ to be a locally finite open cover of M_S that refines $\{U_p:p\in M_S\}$. Hence, $\{\mathscr{B}_W(U):U\in\mathcal{E}_{S,\alpha}\}$ is a refinement of $\mathbb{U}$.

Let us show that $\{\mathscr{B}_W(U):U\in\mathcal{E}_{S,\alpha}\}$ is locally finite. Suppose $A\in\mathscr{B}(W)$ and every neighborhood of A intersects $\bigcup_{U\in\mathcal{E}_{S,\alpha}}\mathscr{B}_W(U)$. Then $\mathscr{B}_W(\mathscr{F}_W(S))$, which contains $\bigcup_{U\in\mathcal{E}_{S,\alpha}}\mathscr{B}_W(U)$ and is closed by Lemma 2.6, must also contain A. Let

V be a neighborhood of $A@S$ that only intersects finitely many members of $\mathcal{E}_{S,\alpha}$. Then $\mathscr{B}_W(V)$ only intersects finitely many elements of $\{\mathscr{B}_W(U) : U \in \mathcal{E}_{S,\alpha}\}$.

By the definition of $\mathcal{B}_\alpha$ and the containment of $\mathcal{A}_\alpha$ in $\mathcal{B}_\alpha$, we have (1) and (2) if $\alpha + 1$ replaces α. By the definition of $\mathcal{A}_\alpha$, we have (5) if $\alpha + 1$ replaces α. By what we have already shown about $\mathcal{E}_{S,\alpha}$ for each $S \in \mathcal{A}_\alpha$, we have (3) and (4) if $\alpha + 1$ replaces α. If λ is 0 or a limit ordinal, and (1)-(5) are satisfied for all $\alpha < \lambda$, then clearly (1)-(5) are satisfied if $\alpha = \lambda$. Thus, by induction, (1)-(5) are satisfied for every ordinal α.

For each $\alpha \in \mathrm{O}n$, define $\mathbb{V}_\alpha$ as follows:

$$\mathbb{V}_\alpha = \bigcup_{\beta<\alpha} \bigcup_{S\in\mathcal{A}_\beta} \{\mathscr{B}_W(U) : U \in \mathcal{E}_{S,\beta}\}. \tag{3.1}$$

By (3), all elements of $\mathbb{V}_\alpha$ are open. By (4), $\mathbb{V}_\alpha$ is a refinement of $\mathbb{U}$. Let us show that $\mathbb{V}_\alpha$ is locally finite. Suppose $\beta, \gamma < \alpha$ and $S \in \mathcal{A}_\beta$ and $T \in \mathcal{A}_\gamma$ and $U \in \mathcal{E}_{S,\beta}$ and $V \in \mathcal{E}_{T,\gamma}$ and $\mathscr{B}_W(U) \cap \mathscr{B}_W(V) \neq \emptyset$. Then $\mathscr{B}_W(\mathscr{F}_W(S))$ is not disjoint from $\mathscr{B}_W(\mathscr{F}_W(T))$; hence, $S \subseteq T$ or $T \subseteq S$, for W is a tree. If $S \subsetneq T$, then $T@S \notin U$ by (5); whence,

$$\emptyset = \mathscr{B}_W(U) \cap \mathscr{B}_W(\{T@S\}) \supseteq \mathscr{B}_W(U) \cap \mathscr{B}_W(\mathscr{F}_W(T)) \supseteq \mathscr{B}_W(U) \cap \mathscr{B}_W(V)\,,$$

which is absurd. Therefore, $T \subseteq S$. By symmetry, $S \subseteq T$; hence, $S = T$. Hence, by (1), $\beta = \gamma$. Therefore, the unions in (3.1) are disjoint unions. Moreover, by (4), the set $\{\mathscr{B}_W(U) : U \in \mathcal{E}_{S,\beta}\}$ is locally finite for all $\beta < \alpha$ and $S \in \mathcal{A}_\beta$. Hence, $\mathbb{V}_\alpha$ is locally finite. Therefore, it suffices to prove that $\mathbb{V}_\alpha$ covers $\mathscr{B}(W)$ for some ordinal α.

Let $A \in \mathscr{B}(W)$. Suppose there exists $S \in \mathscr{S}(W) \setminus \bigcup_{\alpha\in\mathrm{O}n} \mathcal{A}_\alpha$ such that $S \subseteq A$. Choose S to be as small as possible. Then there exists $\alpha \in \mathrm{O}n$ such that S is a $\subseteq$-minimal element of $\mathscr{S}(W) \setminus \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Therefore, by definition of $\mathcal{A}_\alpha$, there exist $\beta < \alpha$ and $T \in \mathcal{A}_\beta$ such that $T \subsetneq S$ and $A@T = S@T \in \bigcup \mathcal{E}_{T,\beta}$. Therefore, $A \in \bigcup \mathbb{V}_\alpha$.

Suppose every proper semibranch contained in A is in $\bigcup_{\alpha\in\mathrm{O}n} \mathcal{A}_\alpha$. Further suppose A has no maximum. Since $A \in \bigcup \mathbb{U}$, there exist $R \in \mathscr{S}(W)$ and $U \in \mathcal{T}_R$ and $\mathcal{U} \in \mathbb{U}$ such that $A@R \in U$ and $\mathscr{B}_W(U) \subseteq \mathcal{U}$. Set $S = R \cup \{A@R\}$. Then $S \in \mathcal{A}_\alpha$ for some $\alpha \in \mathrm{O}n$. Moreover, by definition, $\mathcal{E}_{S,\alpha} = \{\mathscr{F}_W(S)\}$. Set $T = S \cup \{A@S\}$. Then $T@S \in \mathscr{F}_W(S) = \bigcup \mathcal{E}_{S,\alpha}$; hence, by (5), $T \notin \bigcup_{\beta\in\mathrm{O}n} \mathcal{A}_\beta$, which is absurd. Therefore, A has a maximum. Set $S = A \setminus \{\max A\}$. Then there exists $\alpha \in \mathrm{O}n$ such that $S \in \mathcal{A}_\alpha$. By (3), $A@S \in \bigcup \mathcal{E}_{S,\alpha}$; hence, $A \in \bigcup \mathbb{V}_{\alpha+1}$.

Thus, for each $A \in \mathscr{B}(W)$, there exists $\xi(A) \in \mathrm{O}n$ such that $A \in \bigcup \mathbb{V}_{\xi(A)}$. Therefore, $\mathscr{B}(W) = \bigcup \mathbb{V}_\xi$ where $\xi = \sup\{\xi(A) : A \in \mathscr{B}(W)\}$. □

The arguments used to prove the above theorem about paracompactness also apply to topological dimension.

Theorem 3.5. *Let $n < \omega$. Suppose all fork spaces of W are nonjagged and have topological dimension at most n. Then $\mathscr{B}(W)$ has topological dimension at most n.*

Proof. The argument is the same as in the proof of Theorem 3.4. Wherever they are mentioned, simply respectively replace paracompactness, local finiteness, and finiteness with having topological dimension at most n, having order at most $n+1$, and having cardinality at most $n+1$. □

In the special case of nonjaggedness where W is a tree with no maximal elements, the branch product topology collapses in the sense that it is independent of the fork topologies. Thus, even assuming all forks are nonjagged, $\mathscr{B}(W)$ may have lower topological dimension than all of its fork spaces.

Theorem 3.6. *The branch product topology on $\mathscr{B}(W)$ is independent of the topologies of the fork spaces which have no elements maximal in W.*

Proof. Let $\mathcal{A}$ be the set of $S \in \mathscr{S}(W)$ such that $\mathscr{F}_W(S)$ has no elements maximal in W. Fix the topologies of all fork spaces except $\mathscr{F}_W(S)$ for $S \in \mathcal{A}$. Then the coarsest possible branch product topology of $\mathscr{B}(W)$ is induced by giving $\mathscr{F}_W(S)$ the indiscrete topology for each $S \in \mathcal{A}$. Likewise, the finest possible branch product topology of $\mathscr{B}(W)$ is induced by giving $\mathscr{F}_W(S)$ the discrete topology for each $S \in \mathcal{A}$. Let $S \in \mathcal{A}$ and $E \subseteq \mathscr{F}_W(S)$. Since W is a tree and no elements of $\mathscr{F}_W(S)$ are maximal in W, we have

$$\mathscr{B}_W(E) = \bigcup_{p \in E} \mathscr{B}_W(\mathscr{F}_W(S \cup \{p\}))\,.$$

Hence, $\mathscr{B}_W(E)$ is open in $\mathscr{B}(W)$ even if $\mathscr{F}_W(S)$ is given the indiscrete topology for each $S \in \mathcal{A}$. Thus, the coarsest possible branch product topology of $\mathscr{B}(W)$ is finer than the finest. Therefore, the branch product topology of $\mathscr{B}(W)$ does not depend on the topology of $\mathscr{F}_W(S)$ for $S \in \mathcal{A}$. □

Corollary 3.7. *Suppose W is a tree with no maximal elements. Then the branch product topology on $\mathscr{B}(W)$ is independent of the topologies of the fork spaces. Moreover, $\mathscr{B}(W)$ is Hausdorff and has topological dimension* 0 *(and is therefore paracompact).*

Proof. The first statement of the corollary immediately follows from Theorem 3.6. Let us prove the second statement. Every fork space is nonjagged because it has no maximal elements. Moreover, we may assume every fork space is discrete, and hence is Hausdorff and has topological dimension 0. By Theorems 2.2 and 3.5, $\mathscr{B}(W)$ is paracompact Hausdorff and has topological dimension 0. □

Remark 3.8. A T_1 space with topological dimension 0 is easily shown to be zero-dimensional in the sense of having a basis of clopen sets.

The next example shows that Corollary 3.7 may not be extended to conclude $\mathscr{B}(W)$ is locally compact or perfectly normal.

Example 3.9. Let $W = \bigcup_{\alpha<\omega_1} \left(\{a_\alpha\} \cup \{b_{\alpha,n} : n < \omega\}\right)$ and let

$$a_\beta > a_\alpha < b_{\alpha,n} < b_{\alpha,n+1} \text{ and } b_{\alpha,0} \not\leq a_\beta \tag{3.2}$$

for all $\alpha < \beta < \omega_1$ and $n < \omega$. The relations in (3.2) uniquely define the ordering of W. Moreover, W is a tree without maximal elements, and Corollary 3.7 applies. However, the singleton $\{\{a_\alpha : \alpha < \omega_1\}\}$ is closed but is not a G_δ set in $\mathscr{B}(W)$; hence, $\mathscr{B}(W)$ is not perfectly normal; hence, $\mathscr{B}(W)$ is not metrizable. Moreover, $\mathscr{B}(W)$ is not locally compact: if $\alpha < \omega_1$, then

$$\left\{\mathscr{B}_W(\{b_{\alpha+n,0}\}) : n < \omega\right\} \cup \left\{\mathscr{B}_W(\{a_{\alpha+\omega}\})\right\}$$

is an open cover of the clopen set $\mathscr{B}_W(\{a_\alpha\})$, and it has no finite subcover; hence, no neighborhood of the branch $\{a_\alpha : \alpha < \omega_1\}$ has compact closure.

4. Branch Product Iterations

Before defining iterated branch products, we need some additional notation. Henceforth, let X_p be a nonempty poset and let $\mathcal{E}_p \subseteq \mathcal{P}(X_p)$ satisfy $\mathscr{B}_{X_p}(\bigcup \mathcal{E}_p) = \mathscr{B}(X_p)$ for each $p \in Y$. Assume that $X_p \cap X_q = \emptyset$ for all distinct $p, q \in Y$. Let Z be the ordered sum $\sum_{p\in Y} X_p$ and let $\mathcal{E} = \bigcup_{p\in Y} \mathcal{E}_p$.

Definition 4.1. We say that $\mathscr{B}(Z, \mathcal{E})$ is the *iterated branch product* of the spaces $\mathscr{B}(X_p, \mathcal{E}_p)$ for which $p \in Y$.

Since $\mathscr{B}(X_p, \mathcal{E}_p)$ could be an iterated branch product for any $p \in Y$, we can iterate branch products arbitrarily many times, justifying the term "iterated." Also, $\mathscr{B}(Z, \mathcal{E})$ is the $\mathcal{E}$-induced branch product by definition. Moreover, if Y and X_p are well-founded for all $p \in Y$, then $\mathscr{B}(Z)\, E$ can be defined in terms of branch products induced by fork spaces, as shown below.

Lemma 4.2. *Suppose Y and X_p are well-founded for all $p \in Y$. Then, for each $S \in \mathscr{S}(Z)$, there is a unique $y \in Y$ such that*

$$\mathscr{F}_Z(S) = \mathscr{F}_{X_y}(S \cap X_y)\,,$$

or there is a unique $I \subseteq Y$ such that $\mathscr{F}_Z(S) = \bigcup_{p\in I} \mathscr{F}_{X_p}(\emptyset)$. Moreover, if both such a y and I exist, then $I = \{y\}$ and $S \cap X_y = \emptyset$.

Proof. Let $S \in \mathscr{S}(Z)$. Let $p \in \mathscr{F}_Z(S)$ and let $y \in Y$ satisfy $p \in X_y$. Suppose p is not minimal in X_y. Then $S \cap X_y$ is cofinal in S; whence, $\mathscr{F}_{X_y}(S \cap X_y) = \mathscr{F}_Z(S)$. Moreover, there is at most one y such that $S \cap X_y$ is cofinal in S, for Z is an ordered sum of $\{X_r : r \in Y\}$. Suppose p is minimal in X_y. Let $q \in \mathscr{F}_Z(S)$ be arbitrary and let $z \in Y$ satisfy $q \in X_z$. Suppose q is not minimal in X_z. Then we have

$$\mathscr{F}_{X_z}(S \cap X_z) = \mathscr{F}_Z(S)\,;$$

hence, $p \in X_z$; hence, $y = z$. Since q is not minimal in X_z but is a minimal strict upper bound of S, there exists $r \in S \cap X_z$. Therefore, p is not a strict upper bound of S, for p is minimal in X_z. This situation is absurd. Therefore, q is minimal in X_z. Thus, there exists $I \subseteq Y$ such that $\mathscr{F}_Z(S) \subseteq \bigcup_{y\in I} \mathscr{F}_{X_y}(\emptyset)$.

Since $X_u \cap X_v = \emptyset$ for all distinct $u, v \in Y$, there is a smallest such I, and hence we may choose I to be as small as possible. Then $\mathscr{F}_{X_w}(\emptyset) \cap \mathscr{F}_Z(S) \neq \emptyset$ for all $w \in I$. Suppose $w \in I$ and $r \in \mathscr{F}_{X_w}(\emptyset) \cap \mathscr{F}_Z(S)$ and $s \in \mathscr{F}_{X_w}(\emptyset) \setminus \mathscr{F}_Z(S)$. Since r is minimal in X_w and is a strict upper bound of S, every element of X_w, including s, is a strict upper bound of S. Since s is not a minimal strict upper bound of S, there exists $t \in Z$ such that t is less than s and is a strict upper bound of S. Since s is minimal in X_w, there exists $v \in Y$ such that $t \in X_v$ and v is less than w. But then t is also less than r, which is absurd. Therefore, $\mathscr{F}_Z(S) = \bigcup_{y\in I} \mathscr{F}_{X_y}(\emptyset)$. Moreover, for any J such that $\mathscr{F}_Z(S) = \bigcup_{y\in J} \mathscr{F}_{X_y}(\emptyset)$, we have $I = J$ because $X_u \cap X_v = \emptyset$ for all distinct $u, v \in Y$.

Finally, suppose $y \in Y$ and $I \subset Y$ satisfy

$$\mathscr{F}_Z(S) = \mathscr{F}_{X_y}(S \cap X_y) = \bigcup_{p\in I} \mathscr{F}_{X_p}(\emptyset)\,.$$

Then clearly $I = \{y\}$. Hence, $\mathscr{F}_{X_y}(S \cap X_y) = \mathscr{F}_{X_y}(\emptyset)$. Finally, $S \cap X_y = \emptyset$ because forks of nonempty proper semibranches of X_y do not contain minimal elements of X_y. □

Definition 4.3. Suppose Y and X_p are well-founded for each $p \in Y$. For each $p \in Y$ and $S \in \mathscr{S}(X_p)$, let $\mathcal{T}_{p,S}$ be a topology of $\mathscr{F}_{X_p}(S)$. For each $S \in \mathscr{S}(Z)$, if $\mathscr{F}_Z(S) = \mathscr{F}_{X_p}(S \cap X_p)$ for some $p \in Y$, then set $\mathcal{O}_S = \mathcal{T}_{p,S\cap X_p}$. If $\mathscr{F}_Z(S) = \bigcup_{p\in E} \mathscr{F}_{X_p}(\emptyset)$ for some $E \subseteq Y$, then let $\mathcal{O}_S$ be the sum topology on $\mathscr{F}_Z(S)$ induced by $\langle \mathcal{T}_{p,\emptyset}\rangle_{p\in E}$. By Lemma 4.2, it is legitimate to define the *sum branch product topology* of $\mathscr{B}(Z)$ to be the branch product topology of $\mathscr{B}\big(Z, \langle \mathcal{O}_S\rangle_{S\in\mathscr{S}(X)}\big)$.

Theorem 4.4. *Supose Y and X_p are well-founded for each $p \in Y$. Further suppose that $\mathcal{E}_p = \bigcup_{S\in\mathscr{S}(X_p)} \mathcal{T}_{p,S}$. Then the topology of $\mathscr{B}(Z, \mathcal{E})$ is the sum branch product topology of $\mathscr{B}(Z)$.*

Proof. For each $S \in \mathscr{S}(Z)$ and $U \in \mathcal{O}_S$, the set U is a union of elements of $\mathcal{E}$ by definition. Thus, $\mathscr{B}_Z(U)$ is open in $\mathscr{B}(Z, \mathcal{E})$. Conversely, suppose $p \in Y$ and $U \in \mathcal{E}_p$. Then there exists $S \in \mathscr{S}(X_p)$ such that $U \in \mathcal{T}_{p,S}$. Let $T \in \mathscr{S}(Y)$ satisfy $p \in \mathscr{F}_Y(T)$. For each $q \in T$, choose $A_q \in \mathscr{B}(X_q)$. Suppose $S \neq \emptyset$. Then $U \in \mathcal{O}_R$ for any $R \in \mathscr{S}(Z)$ for which $S = R \cap X_p$. Such an R always exists: an example is $S \cup \bigcup_{q\in T} A_q$. Thus, $\mathscr{B}_Z(U)$ is open in the sum branch product topology. Suppose $S = \emptyset$. Set $R = \bigcup_{q\in T} A_q$. Then $R \in \mathscr{S}(Z)$ and $\mathscr{F}_Z(R) = \bigcup_{q\in\mathscr{F}_Y(T)} \mathscr{F}_{X_q}(\emptyset)$. Therefore, $U \in \mathcal{T}_{p,\emptyset} \subseteq \mathcal{O}_R$. Thus, $\mathscr{B}_Z(U)$ is open in the sum branch product topology. Thus, the topology of $\mathscr{B}(Z, \mathcal{E})$ is the sum branch product topology of $\mathscr{B}(Z)$, for each topology has a subbasis of open sets also open in the other topology. □

Unlike branch products, branch product iterations preserve all the separation axioms preserved by sums and products. This is because, up to homeomorphism, branch product iterations can be constructed using the familiar topological operations of products, sums, and subspaces, as shown by the next theorem.

Lemma 4.5. *Suppose $p \in Y$. Then $\mathscr{B}_Z(X_p)$ is clopen*

Proof. Suppose $A \in \mathscr{B}(Z) \setminus \mathscr{B}_Z(X_p)$. Then $\pi(A) \in \mathscr{B}(Y) \setminus \mathscr{B}_Y(\{p\})$. Since $\pi(A)$ is a maximal chain, there exists $q \in \pi(A)$ such that p and q are incomparable. Thus, $A \in \mathscr{B}_Z(X_q)$, which is open in $\mathscr{B}(Z, \mathcal{E})$ and disjoint from $\mathscr{B}_Z(X_p)$. □

Theorem 4.6. *Let $\mathcal{C}$ be a topological class hereditary with respect to subspaces and closed under finite sums and products of $|Y|$-many spaces. Suppose $\mathscr{B}(X_p, \mathcal{E}_p) \in \mathcal{C}$ for all $p \in Y$. Then $\mathscr{B}(Z, \mathcal{E}) \in \mathcal{C}$.*

Proof. We construct an element of $\mathcal{C}$ homeomorphic to $\mathscr{B}(Z, \mathcal{E})$. Let Λ be a nonempty member of $\mathcal{C}$, such as $\mathscr{B}(X_p, \mathcal{E}_p)$ for some $p \in Y$. Fix $\lambda \in \Lambda$. For each $p \in Y$, let M_p be the topological sum of $\mathscr{B}(X_p, \mathcal{E}_p)$ and Λ. Given a partial function f on Y, let $\zeta(f)$ be the unique extension of f to a complete function on Y for which $\zeta(f)(p) = \lambda$ for all $p \in Y \setminus \operatorname{dom} f$. Set $M = \prod_{p\in Y} M_p$; set

$$N = \bigcup_{A\in\mathscr{B}(Y)} \left\{ \zeta(f) : f \in \prod_{p\in A} \mathscr{B}(X_p) \right\}$$

and give it the subspace topology induced by M. Then $N \in \mathcal{C}$.

Let us show that $\mathscr{B}(Z, \mathcal{E})$ is homeomorphic to N. For each $A \in \mathscr{B}(Z)$, let $\eta(A)$ denote the partial function on Y for which $\eta(A)(p) = A \cap X_p$ for all $p \in Y$ for which $A \cap X_p \neq \emptyset$ and $\eta(A)(p)$ is undefined otherwise. Set $\theta = \zeta \circ \eta$. Then θ is a bijection from $\mathscr{B}(Z, \mathcal{E})$ to N because $\operatorname{dom} \eta(A) \in \mathscr{B}(Y)$ for all $A \in \mathscr{B}(Z)$.

Let us show that θ is a homeomorphism. Let $\mathcal{O}$ denote the topology of Λ. Set $\mathcal{O}_p = \mathcal{O} \cup \{\mathscr{B}_{X_p}(\mathcal{E}) : E \in \mathcal{E}_p\}$. Then $\mathcal{O}_p$ is a subbasis for M_p. Hence, $\bigcup_{p\in Y}\{\{f \in N : f(p) \in U\} : U \in \mathcal{O}_p\}$ is a subbasis for N. Suppose $p \in Y$ and $U \in \mathcal{O}_p$. If $U = \mathscr{B}_{X_p}(E)$ for some $E \in \mathcal{E}_p$, then $\theta^{-1}\big(\{f \in N : f(p) \in U\}\big) = \mathscr{B}_Z(E)$, which is open. If $\lambda \in U \in \mathcal{O}$, then $\theta^{-1}\big(\{f \in N : f(p) \in U\}\big) = \mathscr{B}(Z) \setminus \mathscr{B}_Z(X_p)$, which is open by Lemma 4.5. If $\lambda \notin U \in \mathcal{O}$, then $\{f \in N : f(p) \in U\} = \emptyset$. Thus, θ is continuous. Furthermore, θ is open because $\theta(\mathscr{B}_Z(E)) = \{f \in N : f(p) \in \mathscr{B}_{X_p}(E)\}$ for all $E \in \mathcal{E}_p$ and $\{\mathscr{B}_Z(E) : E \in \mathcal{E}\}$ is a subbasis for $\mathscr{B}(Z, \mathcal{E})$. Thus, θ is a homeomorphism. □

Corollary 4.7. *The T_0 axiom, the T_1 axiom, the T_2 axiom, regularity, complete regularity, total disconnectedness, and the property of having a basis of clopen subsets are each preserved by branch product iterations. Moreover, if Y is countable and $\mathscr{B}(X_p, \mathcal{E}_p)$ is metrizable for all $p \in Y$, then $\mathscr{B}(Z, \mathcal{E})$ is metrizable.*

Corollary 4.7 mentions the most commonly used topological properties that are preserved by sums and products. Since these properties are all also hereditary with respect to subspaces, the following example of a topological class closed under products and sums but not under branch product iterations is necessarily a bit obscure.

Definition 4.8. For any set E, let $[E]^1$ denote the set of singleton subsets of E.

Example 4.9. Let $\mathcal{D}$ denote the closure of the class of discrete spaces with respect to sums and products. A nondiscrete space in $\mathcal{D}$ must be constructed using infinite products; hence, such a space has cardinality at least $2^{\aleph_0}$.

Let us construct a countable nondiscrete space that is a branch product iteration of discrete spaces. Let Y be the unique poset satisfying $Y = \{a_n, b_n : n < \omega\}$ and $\mathscr{S}(Y) = \{\{a_m : m < n\} : n < \omega\}$ and $\mathscr{F}_Y(\{a - m : m < n\}) = \{a_n, b_n\}$. For each $p \in Y$, let X_p be a singleton. Then $\mathscr{B}(Z, \mathcal{E})$ is homeomorphic to $\mathscr{B}\big(Y, [Y]^1\big)$. Moreover, $\mathscr{B}(Y)$ is countable and $\mathscr{B}\big(Y, [Y]^1\big)$ is not discrete because the branch $\{a_n : n, \omega\}$ is not isolated. Thus, branch product iterations strictly generalize sums and products.

Note that $\mathscr{B}\big(Y, [Y]^1\big)$ is homeomorphic to $\mathscr{B}(Y)$ with the branch product topology induced by giving each fork the discrete topology. Thus, there is a branch product not in $\mathcal{D}$ but induced by fork spaces all in $\mathcal{D}$. Hence, branch products also strictly generalize sums and products.

5. The Tychonoff Theorem

The Tychonoff Theorem has a very nice generalization in terms of branch product iterations. The proof is similar to that of the usual Tychonoff Theorem.

Theorem 5.1. *The space $\mathscr{B}(Z, \mathcal{E})$ is compact if and only if $\mathscr{B}\big(Y, [Y]^1\big)$ is compact and $\mathscr{B}(X_p, \mathcal{E}_p)$ is compact for all $p \in Y$.*

Proof. Let $\pi : \mathscr{B}(Z) \to \mathcal{P}(Y)$ be given by $\pi(A) = \{p \in Y : A \cap X_p \neq \emptyset\}$. It is clear that the range of π is exactly $\mathscr{B}(Y)$.

Suppose $\mathscr{B}\big(Y, [Y]^1\big)$ is not compact. By the Alexander Subbasis Theorem, there exists $I \subseteq Y$ such that $\mathscr{B}_Y(I) = \mathscr{B}(Y)$ but $\mathscr{B}_Y(F) \subsetneq \mathscr{B}(Y)$ for all finite $F \subseteq I$. Set $\mathbb{J} = \{\mathscr{B}_Z(X_p) : p \in I\}$. Then $\mathbb{J}$ is an open cover of $\mathscr{B}(Z, \mathcal{E})$: if $A \in \mathscr{B}(Z)$, then $\pi(A) \in \mathscr{B}(Y)$; whence, $\pi(A) \cap I \neq \emptyset$; whence, $A \cap X_p \neq \emptyset$ for some $p \in I$; whence,

$A \in \bigcup \mathbb{J}$. Suppose F is a finite subset of I. Then there exists $B \in \mathscr{B}(Y) \setminus \mathscr{B}_Y(F)$. For each $p \in B$, choose $A_p \in \mathscr{B}(X_p)$. Then $\bigcup_{p\in B} A_p$ is a branch of Z not in $\bigcup_{p\in F} \mathscr{B}_Z(X_p)$. Therefore, $\mathbb{J}$ has no finite subcover; hence, $\mathscr{B}(Z,\mathcal{E})$ is not compact.

Suppose $\mathscr{B}(X_p,\mathcal{E}_p)$ is not compact for some $p \in Y$. By the Alexander Subbasis Theorem, there exists $\mathcal{A} \subseteq \mathcal{E}_p$ such that $\{\mathscr{B}_{X_p}(I) : I \in \mathcal{A}\}$ is a cover of $\mathscr{B}(X_p)$ with no finite subcover. Consequently, $\{\mathscr{B}_Z(I) : I \in \mathcal{A}\}$ is a cover of $\mathscr{B}_Z(X_p)$ with no finite subcover. Thus, to prove that $\mathscr{B}(Z,\mathcal{E})$ is not compact, it suffices to note that $\mathscr{B}_Z(X_p)$ is closed in $\mathscr{B}(Z,\mathcal{E})$ by Lemma . refbranchiteratesingletonclosedLEM.

Suppose $\mathscr{B}\big(Y,[Y]^1\big)$ is compact and $\mathscr{B}(X_p,\mathcal{E}_p)$ is compact for all $p \in Y$. Further suppose $\mathcal{H} \subseteq \mathcal{E}$ and $\bigcup_{I\in\mathcal{H}} \mathscr{B}_Z(I) = \mathscr{B}(Z)$. By the Alexander Subbasis Theorem, it suffices to prove that $\mathcal{H}$ has a finite subset $\mathcal{H}'$ such that $\bigcup_{I\in\mathcal{H}'} \mathscr{B}_Z(I) = \mathscr{B}(Z)$. For each $p \in Y$, set $\mathcal{H}_p = \mathcal{H} \cap \mathcal{P}(X_p)$. Define J as follows:

$$J = \Big\{p \in Y : \bigcup_{I\in\mathcal{H}_p} \mathscr{B}_{X_p}(I) = \mathscr{B}(X_p)\Big\}.$$

Suppose $\mathscr{B}_Y(J) = \mathscr{B}(Y)$. Since $\mathscr{B}\big(Y,[Y]^1\big)$ is compact, there is a finite set $F \subseteq J$ such that $\mathscr{B}_Y(F) = \mathscr{B}(Y)$. For each $p \in F$, there is a finite set $\mathcal{H}'_p \subseteq \mathcal{H}_p$ such that $\bigcup_{I\in\mathcal{H}'_p} \mathscr{B}_{X_p}(I) = \mathscr{B}(X_p)$, by compactness. Now let A be an arbitrary branch of Z. Then $\pi(A)$ contains an element of F, say p. Then $A \cap X_p$, which is a branch of X_p, intersects an element of $\mathcal{H}'_p$. Set $\mathcal{H}' = \bigcup_{p\in F} \mathcal{H}'_p$. Then $\mathcal{H}'$ has the desired properties.

Suppose $\mathscr{B}_Y(J) \subsetneq \mathscr{B}(Y)$. Then there exists $B \in \mathscr{B}(Y)$ such that $B \cap J = \emptyset$. Hence, for each $p \in B$, there exists a branch A_p of X_p that is not in $\bigcup_{I\in\mathcal{H}_p} \mathscr{B}_{X_p}(I)$. Set $A = \bigcup_{p\in B} A_p$. Then $A \in \mathscr{B}(Z) \setminus \bigcup_{p\in B} \bigcup_{I\in\mathcal{H}_p} \mathscr{B}_Z(I)$. Since $\mathcal{E} \subseteq \bigcup_{p\in Y} \mathcal{P}(X_p)$, we have $\mathcal{H} = \bigcup_{p\in Y} \mathcal{H}_p$. Therefore, $A \notin \bigcup_{I\in\mathcal{H}} \mathscr{B}_Z(I)$, a contradiction. Thus, $\mathscr{B}(Z,\mathcal{E})$ is compact. □

Corollary 5.2 (Tychonoff Theorem)**.** *Suppose I is a set and X_i is a topological space for each $i \in I$. Then $\prod_{i\in I} X_i$ is compact if and only if X_i is compact for all $i \in I$.*

Proof. We may assume that $I \neq \emptyset$ and $X_i \neq \emptyset$ for all $i \in I$. We may further assume that $X_i \cap X_j = \emptyset$ for all distinct $i, j \in I$. For each $i \in I$, let $\mathcal{O}_i$ be the topology of X_i. Give I an arbitrary linear ordering and, for each $i \in I$, order X_i such that it is an antichain. Then $\mathscr{B}(X_i,\mathcal{O}_i)$ is homeomorphic to X_i. Set $P = \sum_{i\in I} X_i$. Then $\mathscr{B}\big(P,\bigcup_{i\in I}\mathcal{O}_i\big)$ is homeomorphic to $\prod_{i\in I} X_i$. Since $\mathscr{B}(I)$ is a singleton, $\mathscr{B}\big(I,[I]^1\big)$ is compact. Thus, by Theorem 5.1, $\mathscr{B}\big(P,\bigcup_{i\in I}\mathcal{O}_i\big)$ is compact if and only if $\mathscr{B}(X_i,\mathcal{O}_i)$ is compact for all $i \in I$. Thus, $\prod_{i\in I} X_i$ is compact if and only if X_i is compact for all $i \in I$. □

Theorem 5.1 has an analogue for sum branch product topologies.

Theorem 5.3. *Supose Y and X_p are well-founded for each $p \in Y$. Further suppose that $\mathcal{E}_p = \bigcup_{S\in\mathscr{S}(X_p)} \mathcal{T}_{p,S}$. Then $\mathscr{B}(Z,\mathcal{E})$ with the sum branch product topology is compact if and only if the space*

$$\mathscr{B}\big(Y,\langle\mathcal{P}(\mathscr{F}_Y(S))\rangle_{S\in\mathscr{S}(Y)}\big)$$

is compact and $\mathscr{B}\big(X_p,\langle\mathcal{T}_{p,S}\rangle_{S\in\mathscr{S}(X_p)}\big)$ is compact for all $p \in Y$.

Proof. By definition, $\mathscr{B}\big(X_p, \langle \mathcal{T}_{p,S}\rangle_{S\in\mathscr{S}(X_p)}\big)$ is equal to $\mathscr{B}(X_p, \mathcal{E}_p)$. Moreover, the space

$$\mathscr{B}\big(Y, \langle \mathcal{P}(\mathscr{F}_Y(S))\rangle_{S\in\mathscr{S}(Y)}\big)$$

is clearly equal to $\mathscr{B}\big(Y, [Y]^1\big)$. Thus, by Theorem 5.1, it suffices to note that, by Theorem 4.4, $\mathscr{B}\Big(Z, \bigcup_{p\in Y}\mathcal{E}\Big)$ is just $\mathscr{B}(Z)$ with the sum branch product topology. □

By the same argument as the proof of Corollary 5.2, when Y is a chain, Theorem 5.1 reduces to the Tychonoff Theorem, for in this case, $\mathscr{B}(Y)$ is a singleton and $\mathscr{B}(Z,\mathcal{E})$ is easily checked to be homeomorphic to $\prod_{p\in Y}\mathscr{B}(X_p,\mathcal{E}_p)$. It is natural to ask which posets P besides chains are such that $\mathscr{B}\big(P,[P]^1\big)$ is compact. By the Alexander Subbasis Theorem, $\mathscr{B}\big(P,[P]^1\big)$ is compact if and only if, for every $E \subseteq P$ such that $\mathscr{B}_P(E) = \mathscr{B}(P)$, there exists a finite set $F \subseteq E$ such that $\mathscr{B}_P(F) = \mathscr{B}(P)$. This characterization has the advantage of conciseness, but it gives us very little direct information about P. For well-founded posets, next theorem remedies this deficiency.

Definition 5.4. Given $S \in \mathscr{S}(X)$, we say that S is *capped* if S contains an element p such that $\mathscr{B}_X(\{p\}) \subseteq \mathscr{B}_X(\mathscr{F}_X(S))$.

Remark 5.5. If $S \in \mathscr{S}(X)$ and S has a maximum, then the minimal strict upper bounds of $\max S$ are exactly the minimal strict upper bounds of S; whence, $\mathscr{B}_X(\{\max S\}) \subseteq \mathscr{B}_X(\mathscr{F}_X(S))$; whence, S is capped. Thus, whether a proper semibranch is capped is only an interesting question when it has no maximum.

Lemma 5.6. *Let P be a nonempty poset. Then $\mathscr{B}_P(\{p\})$ is clopen in $\mathscr{B}\big(P,[P]^1\big)$ for all $p \in P$.*

Proof. Simply note that $\mathscr{B}_P(\{p\}) = \mathscr{B}(P) \setminus \bigcup_{p\not\leq q\not\leq q}\mathscr{B}_P(\{q\})$. □

Theorem 5.7. *The space $\mathscr{B}\big(X,[X]^1\big)$ is compact if and only if every fork in X is finite and every nonempty proper semibranch of X is capped.*

Proof. Suppose $\mathscr{B}\big(X,[X]^1\big)$ is compact. Let $S \in \mathscr{S}(X)$. Let $\mathcal{A} = \bigcap_{p\in S}\mathscr{B}_X(\{p\})$. By Lemma 5.6, $\mathcal{A}$ is closed; hence, it is compact. Moreover, a branch A is in $\mathcal{A}$ if and only if it contains S. Therefore, $\{\mathscr{B}_X(\{p\}) : p \in \mathscr{F}_X(S)\}$ is an open cover of $\mathcal{A}$ with no proper subcover. Hence, $\mathscr{F}_X(S)$ is finite.

Further suppose $S \in \mathscr{S}(X) \setminus \{\emptyset\}$. Let us show that S is capped. Set

$$\mathcal{V}_p = \mathscr{B}_X(\{p\}) \setminus \mathscr{B}_X(\mathscr{F}_X(S))$$

for each $p \in S$. By Lemma 5.6, $\mathcal{V}_p$ is closed for all $p \in S$. Suppose $A \in \bigcap_{p\in S}\mathcal{V}_p$. Then A contains S. Therefore, A intersects $\mathscr{F}_X(S)$; hence, $A \notin \bigcup_{p\in S}\mathcal{V}_p$. This situation is absurd; hence, $\bigcap_{p\in S}\mathcal{V}_p = \emptyset$.

By compactness, there exists a finite set $F \subseteq S$ such that $\bigcap_{p\in F}\mathcal{V}_p = \emptyset$. Since S is a chain, so is F. Suppose $\mathcal{V}_{\max F}$ is nonempty. Then choose $A \in \mathcal{V}_{\max F}$. Set $A' = \{p \in S : p \leq \max F\}$ and $A'' = \{p \in A : p \geq \max F\}$. Then $A' \cup A''$ is a branch of X. Let us show that $A' \cup A'' \in \bigcap_{p\in F}\mathcal{V}_p$. First, $F \subseteq A'$; hence, $A' \cup A'' \in \bigcap_{p\in F}\mathscr{B}_X(\{p\})$. Second, $A' \subseteq S$ and $A'' \subseteq A$ and

$$S \cap \mathscr{F}_X(S) = A \cap \mathscr{F}_X(S) = \emptyset;$$

hence, $A' \cup A'' \notin \mathscr{B}_X(\mathscr{F}_X(S))$. Thus, $A' \cup A'' \in \bigcap_{p \in F} \mathcal{V}_p$, which is absurd. Therefore, $\mathcal{V}_{\max F} = \emptyset$; hence, S is capped. Thus, the "only if" part of the theorem holds.

Let us prove the "if" part. Suppose every fork in X is finite and every nonempty proper semibranch of X is capped. Further suppose $E \subseteq X$ and $\mathscr{B}_X(E) = \mathscr{B}(X)$. Then it suffices to prove that $\mathscr{B}_X(F) = \mathscr{B}(X)$ for some finite set $F \subseteq E$. Suppose that no such F exists. Then it suffices to find a branch A such that $A \cap E = \emptyset$, for this contradicts $\mathscr{B}_X(E) = \mathscr{B}(X)$.

By Proposition 1.5, we may define A by defining $A@A_\alpha$ in terms of A_α for every $\alpha < h(A)$. Suppose $\alpha \in \mathrm{O}n$ and A_α is a proper semibranch such that $\mathscr{B}_X(\{p\}) \nsubseteq \mathscr{B}_X(F)$ for all $p \in A_\alpha$ and for all finite subsets F of E. Suppose, for each $q \in \mathscr{F}_X(A_\alpha)$, that there exists a finite set $F_q \subseteq E$ such that $\mathscr{B}_X(\{q\}) \subseteq \mathscr{B}_X(F_q)$. Set $G = \bigcup_{q \in \mathscr{F}_X(A_\alpha)} F_q$. Then G is finite because $\mathscr{F}_X(A_\alpha)$ is finite. If $A_\alpha = \emptyset$, then $\mathscr{B}(X) = \mathscr{B}_X(\mathscr{F}_X(\emptyset)) \subseteq \mathscr{B}_X(G)$, in contradiction with our assumption that $\mathscr{B}_X(F) \neq \mathscr{B}(X)$ for all finite subsets F of E. Therefore, $A_\alpha \neq \emptyset$; hence, A_α is capped; hence, there exists $p \in A_\alpha$ such that $\mathscr{B}_X(\{p\}) \subseteq \mathscr{B}_X(\mathscr{F}_X(A_\alpha)) \subseteq \mathscr{B}_X(G)$, which is absurd. Therefore, there exists $q \in \mathscr{F}_X(A_\alpha)$ such that $\mathscr{B}_X(\{q\}) \nsubseteq \mathscr{B}_X(F)$ for all finite subsets F of E. Let $A@A_\alpha$ be such a q. Then, by induction, every element p of A is such that $\mathscr{B}_X(\{p\}) \nsubseteq \mathscr{B}_X(F)$ for all finite subsets F of E. Therefore, $p \notin E$ for all $p \in A$; hence, $A \cap E = \emptyset$. $\square$

For a tree W, the compactness of $\mathscr{B}\big(W, [W]^1\big)$ comes close to imposing finiteness of proper semibranches. The next theorem makes this statement more precise. However, the proof of this theorem requires some new notation as well as several lemmas.

Definition 5.8. Given a poset P, let $<'_P$ be the set of ordered pairs $\langle p, q \rangle$ in P^2 for which p is the maximum strict lower bound of q and q is the minimum strict upper bound of p. Let $\sim_P$ be the finest equivalence relation that contains $<'_P$. If P is clear from the context, then $<'_P$ and $\sim_P$ may be respectively abbreviated by $<'$ and $\sim$.

Remark 5.9. For every poset P, all $\sim_P$-classes are clearly chains.

Lemma 5.10. *Suppose P is a poset. Then $\sim_P$ is compatible with the ordering of P.*

Proof. Let $\leq$ denote the ordering of P. Suppose $p, p', q, q' \in P$ and

$$p' \sim p < q \sim q' \not\sim p.$$

It suffices to show that $p' < q'$. To do so, we first show that $p' < q$. Since $p \sim p'$, we have $p \leq p'$ or $p' \leq p$. If $p' \leq p$, then $p' < q$. Suppose $p \leq p' \not< q$. Then $p' \sim p \leq p'$; hence, there exist $n < \omega$ and $r_0, \ldots, r_n \in P$ such that

$$p = r_0 <' r_1 <' \cdots <' r_n = p'.$$

Let m be the minimal ordinal not greater than n such that $r_m \not< q$. Then $m > 0$. Therefore, $r_{m-1} < q$ and r_m is the minimum strict upper bound of r_{m-1}; hence, $q \geq r_m$ and $q \not> r_m$; hence, $q = r_m$; hence, $p \sim q$, which is absurd. Therefore, $p' < q$.

Since $q \sim q'$, we have $q \leq q'$ or $q' \leq q$. If $q \leq q'$, then $p' < q'$. Suppose $p' \not< q' \leq q$. Since $q \sim q' \leq q$, there exist $n < \omega$ and $r_0, \ldots, r_n \in P$ such that

$q' = r_0 <' r_1 <' \cdots <' r_n = q$. Let m be the maximum ordinal not greater than n such that $p' \not< r_m$. Then $m < n$. Therefore, $p' < r_{m+1}$ and r_m is the maximum strict lower bound of r_{m+1}; hence, $p' \leq r_m$ and $p' \not< r_m$; hence, $p' = r_m$; hence, $p' \sim q$, which is absurd. Therefore, $p' < q'$. □

Lemma 5.11. *Suppose P is a poset and $Q = P/\approx$ where $\approx$ is an equivalence relation on P that is compatible with the ordering of P and whose classes are all chains. Then $\mathscr{B}\big(P, [P]^1\big)$ is homeomorphic to $\mathscr{B}\big(Q, [Q]^1\big)$.*

Proof. Define $f\colon \mathscr{B}(P) \to \mathcal{P}(Q)$ by $f(A) = \{p/\approx\colon p \in A\}$ for all $A \in \mathscr{B}(P)$. Since $\approx$ is compatible with the ordering of P, the poset P is an ordered sum of its $\approx$-classes. Since these classes are all chains, every branch of P is a union of $\approx$-classes. Thus, f is a bijection from $\mathscr{B}(P)$ to $\mathscr{B}(Q)$. Let us show that f is a homeomorphism from $\mathscr{B}\big(P, [P]^1\big)$ to $\mathscr{B}\big(Q, [Q]^1\big)$. If $p \in P$ and $A \in \mathscr{B}_P(\{p\})$, then clearly $f(A) \in \mathscr{B}_Q(\{p/\approx\})$. Conversely, if $p \in P$ and $B \in \mathscr{B}_Q(\{p/\approx\})$, then $f^{-1}(B) \in \mathscr{B}_P(\{p\})$, for $f^{-1}(B)$ is a union of $\approx$-classes. Thus, f bijects $\mathscr{B}_P(\{p\})$ onto $\mathscr{B}_Q(\{p/\approx\})$ for each $p \in P$. Hence, f sends a subbasis of $\mathscr{B}\big(P, [P]^1\big)$ to a subbasis of $\mathscr{B}\big(Q, [Q]^1\big)$, and f^{-1} sends a subbasis of $\mathscr{B}\big(Q, [Q]^1\big)$ to a subbasis of $\mathscr{B}\big(P, [P]^1\big)$. □

Lemma 5.12. *Suppose P is a poset. Then there exists a poset Q such that $\sim_Q$ is discrete and $\mathscr{B}\big(P, [P]^1\big)$ is homeomorphic to $\mathscr{B}\big(Q, [Q]^1\big)$. Moreover, if P is a tree, then we may choose Q to be a tree.*

Proof. Set $Q_0 = P$ and let $\approx_0$ denote the equality relation on P. Given $\alpha \in \mathrm{O}n$ and Q_α and an equivalence relation $\approx_\alpha$ such that $Q_\alpha = P/{\approx_\alpha}$, let $\approx_{\alpha+1}$ be the set of ordered pairs $\langle p, q\rangle \in P^2$ such that $p/{\approx_\alpha} \sim_{Q_\alpha} q/{\approx_\alpha}$. Set $Q_{\alpha+1} = P/{\approx_{\alpha+1}}$. Since every $\sim_{Q_\alpha}$-class is a chain, if every $\approx_\alpha$-class is a chain, then so is every $\approx_{\alpha+1}$-class. Moreover, since $\sim_{Q_\alpha}$ is compatible with the ordering of Q_α by Lemma 5.10, if $\approx_\alpha$ is compatible with the ordering of P, then so is $\approx_{\alpha+1}$.

Given a limit ordinal λ and equivalence relations $\langle \approx_\alpha\rangle_{\alpha<\lambda}$, set $\approx_\lambda = \bigcup_{\alpha<\lambda} \approx_\alpha$. Suppose that every $\approx_\alpha$-class is a chain and $\approx_\alpha \subseteq \approx_\beta$ for all $\alpha < \beta < \lambda$. Then $\approx_\lambda$ is an equivalence relation and every $\approx_\lambda$-class is a chain. Set $Q_\lambda = P/{\approx_\lambda}$. Further suppose that $\approx_\alpha$ is compatible with the ordering of P for all $\alpha < \lambda$. Then $\approx_\lambda$ is compatible with the ordering of P: if $p, p', q, q' \in P$ and $p' \approx_\lambda p < q \approx_\lambda q' \not\approx_\lambda p$ where $\leq$ is the ordering of P, then $p' \approx_\alpha p < q \approx_\alpha q' \not\approx_\alpha p$ for some $\alpha < \lambda$; whence, $p' < q'$.

By induction, for all $\alpha \in \mathrm{O}n$, the equivalence relation $\approx_\alpha$ is compatible with the ordering of P and all its equivalence classes are chains. Hence, $\mathscr{B}\big(P, [P]^1\big)$ is homeomorphic to $\mathscr{B}\big(Q_\alpha, [Q_\alpha]^1\big)$ by Lemma 5.11. Since there are at most $\big|\mathcal{P}\big(P^2\big)\big|$ equivalence relations on P, there exists $\alpha \in \mathrm{O}n$ such that $\approx_\alpha = \approx_{\alpha+1}$. Set $Q = Q_\alpha$. Then $\sim_Q$ is discrete and $\mathscr{B}\big(P, [P]^1\big)$ is homeomorphic to $\mathscr{B}\big(Q, [Q]^1\big)$.

Finally, suppose Q is not a tree. Then there exists $q \in Q$ such that the set of lower bounds of q are not well-ordered. Let L denote of the set of lower bounds of q. Suppose L is not a chain. Then there exist $p, p' \in P$ such that $p/{\approx_\alpha}, p'/{\approx_\alpha} \in L$ and $p/{\approx_\alpha}$ is incomparable with $p'/{\approx_\alpha}$; whence, p is incomparable with p'. Moreover, there exists $r \in P$ such that $r/{\approx_\alpha} = q$ and p and p' are lower bounds of r. Thus, P is not a tree. Suppose L is a chain. Then L contains a strictly descending sequence $\langle q_n\rangle_{n<\omega}$. For each $n < \omega$, choose $p_n \in P$ such that $p_n/{\approx_\alpha} = q_n$. Then $\langle p_n\rangle_{n<\omega}$ is strictly descending in P. Thus, P is not a tree. Therefore, Q is a tree if P is a tree. □

Theorem 5.13. *The space $\mathscr{B}\big(W,[W]^1\big)$ is compact if and only if there is a tree V such that $\mathscr{B}\big(W,[W]^1\big)$ is homeomorphic to $\mathscr{B}\big(V,[V]^1\big)$ and all forks and proper semibranches of V are finite.*

Proof. By Lemma 5.12, there is a tree V such that $\sim_V$ is discrete and $\mathscr{B}\big(W,[W]^1\big)$ is homeomorphic to $\mathscr{B}\big(V,[V]^1\big)$. Suppose $\mathscr{B}\big(W,[W]^1\big)$ is compact. Then $\mathscr{B}\big(V,[V]^1\big)$ is also compact. By Theorem 5.7, all forks in V are finite. Suppose $S \in \mathscr{S}(V)$ and S is infinite. Then S_ω has order type ω. By Theorem 5.7, there exists $n < \omega$ such that $\mathscr{B}_V(\{S@S_n\}) \subseteq \mathscr{B}_V(\mathscr{F}_V(S_\omega))$. Suppose there exists $p \in \mathscr{F}_V(S_{n+1})$ such that $p \neq S@S_{n+1}$. Since V is a tree, p is not comparable with any element of $\mathscr{F}_V(S_\omega)$. Therefore, no branch containing p intersects $\mathscr{F}_V(S_\omega)$. Since $S@S_n < p$, there exists a branch containing p and $S@S_n$; hence, $\mathscr{B}_V(\{S@S_n\})$ is not contained in $\mathscr{B}_V(\mathscr{F}_V(S_\omega))$, which is absurd. Therefore, $\mathscr{F}_V(S_{n+1}) = \{S@S_{n+1}\}$. Thus, $S@S_{n+1}$ is the minimum strict upper bound of $S@S_n$. Moreover, since V is a tree, $S@S_n$ is the maximum strict lower bound of $S@S_{n+1}$. Therefore, $S@S_n <' S@S_{n+1}$; hence, $S@S_n \sim S@S_{n+1}$, in contradiction with our choice of V. Therefore, all proper semibranches of V are finite.

Conversely, suppose there is a tree V such that $\mathscr{B}\big(W,[W]^1\big)$ is homeomorphic to $\mathscr{B}\big(V,[V]^1\big)$ and all forks and proper semibranches of V are finite. Then it suffices to show that $\mathscr{B}\big(V,[V]^1\big)$ is compact. By Theorem 5.7, it suffices to show that every nonempty proper semibranch of V is capped. Suppose $S \in \mathscr{S}(V) \setminus \{\emptyset\}$. Then S is finite. Clearly, $\mathscr{B}_V(\{\max S\}) \subseteq \mathscr{B}_V(\mathscr{F}_V(S))$; hence, S is capped. □

A weaker notion of compactness is $\mathcal{D}$-compactness:

Definition 5.14. Let K be a nonempty set and let $\mathcal{D}$ be an ultrafilter on K. Given a topological space M, a map $f\colon K \to M$, and a point $p \in M$, we say p is a *$\mathcal{D}$-limit point* of f if $f^{-1}(U) \in \mathcal{D}$ for every neighborhood U of p. We say M is *$\mathcal{D}$-compact* if every map $f\colon K \to M$ has a $\mathcal{D}$-limit point.

The original definition for $K = \omega$ is due to Bernstein[1]. Bernstein proves that compactness implies $\mathcal{D}$-compactness and that the Tychonoff Theorem still holds if $\mathcal{D}$-compactness replaces compactness. Saks[2] defined $\mathcal{D}$-compactness for arbitrary K and generalized these results.

The Tychonoff Theorem is generalized by Theorem 5.1, and the Tychonoff Theorem for $\mathcal{D}$-compactness is generalized in exactly the same way by the next theorem. The theorem needs the following lemma.

Lemma 5.15. *Suppose M is $\mathcal{D}$-compact space with closed subspace N. Then N is $\mathcal{D}$-compact.*

Proof. Let f be a map from K to N and let p be a $\mathcal{D}$-limit point of f in M. If $p \notin N$, then $M \setminus N$ is a neighborhood of p and $f^{-1}(M \setminus N) = \emptyset$, in contradiction with the definition of $\mathcal{D}$-limit point. Hence, $p \in N$; hence, N is $\mathcal{D}$-compact. □

Theorem 5.16. *The space $\mathscr{B}(Z,\mathcal{E})$ is $\mathcal{D}$-compact if and only if $\mathscr{B}\big(Y,[Y]^1\big)$ is $\mathcal{D}$-compact and $\mathscr{B}(X_p,\mathcal{E}_p)$ is $\mathcal{D}$-compact for all $p \in Y$.*

Proof. Let $\pi\colon \mathscr{B}(Z) \to \mathscr{B}(Y)$ be given by

$$\pi(A) = \{p \in Y : A \cap X_p \neq \emptyset\}$$

for all $A \in \mathscr{B}(Z)$. For each $p \in Y$, let B_p be an arbitrary branch of X_p. Let $\rho\colon \mathscr{B}(Y) \to \mathscr{B}(Z)$ be given by $\rho(A) = \bigcup_{p\in A} B_p$ for all $A \in \mathscr{B}(Y)$.

Suppose $\mathscr{B}\big(Y, [Y]^1\big)$ is not $\mathcal{D}$-compact. Then there is a map $f\colon K \to \mathscr{B}(Y)$ with no $\mathcal{D}$-limit point. In particular, there is no $A \in \mathscr{B}(Y)$ such that $f^{-1}(\mathscr{B}_Y(\{p\}) \in \mathcal{D}$ for all $p \in A$. Let B be an arbitrary branch of Z. Then there exists $p \in \pi(B)$ such that $f^{-1}(\mathscr{B}_Y(\{p\}) \notin \mathcal{D}$. Furthermore, $f^{-1}(\mathscr{B}_Y(\{p\}) = (\rho \circ f)^{-1}(\mathscr{B}_Z(X_p))$ and $\mathscr{B}_Z(X_p)$ is a neighborhood of B. Hence, B is not a $\mathcal{D}$-limit point of $\rho \circ f$. Hence, $\mathscr{B}(Z, \mathcal{E})$ is not $\mathcal{D}$-compact.

Suppose $p \in Y$ and $\mathscr{B}(X_p, \mathcal{E}_p)$ is not $\mathcal{D}$-compact. Then there is a map f from K to $\mathscr{B}(X_p)$ with no $\mathcal{D}$-limit point. Let A be an arbitrary element of $\mathscr{B}_Z(X_p)$. Then, since $\{\mathscr{B}_{X_p}(U) : U \in \mathcal{E}_p\}$ is a subbasis for $\mathscr{B}(X_p, \mathcal{E}_p)$ and $\mathcal{D}$ is a filter, there exists $U \in \mathcal{E}_p$ such that $A \cap X_p \in \mathscr{B}_{X_p}(U)$ and $f^{-1}(\mathscr{B}_{X_p}(U)) \notin \mathcal{D}$. Let g be a map from $\mathscr{B}(X_p)$ to $\mathscr{B}(Z)$ such that $B \subseteq g(B)$ for all $B \in \mathscr{B}(X_p)$ and $g(A \cap X_p) = A$. Then A is not a $\mathcal{D}$-limit point of $g \circ f$, for we have

$$(g \circ f)^{-1}(\mathscr{B}_Z(U)) = f^{-1}(\mathscr{B}_{X_p}(U)) \notin \mathcal{D}.$$

Thus, $\mathscr{B}_Z(X_p)$ is not a $\mathcal{D}$-compact subspace of $\mathscr{B}(Z, \mathcal{E})$. But $\mathscr{B}_Z(X_p)$ is a closed subspace of $\mathscr{B}(Z, \mathcal{E})$ by Lemma 4.5. Thus, $\mathscr{B}(Z, \mathcal{E})$ is not $\mathcal{D}$-compact.

Suppose $\mathscr{B}\big(Y, [Y]^1\big)$ is $\mathcal{D}$-compact and $\mathscr{B}(X_p, \mathcal{E}_p)$ is $\mathcal{D}$-compact for all $p \in Y$. Let f be a map from K to $\mathscr{B}(Z)$. Since $\{\mathscr{B}_Z(U) : U \in \mathcal{E}\}$ is a subbasis for $\mathscr{B}(Z, \mathcal{E})$ and $\mathcal{D}$ is a filter, it suffices to show that Z has a branch B such that, for every $U \in \mathcal{E}$, if $B \in \mathscr{B}_Z(U)$, then $f^{-1}(\mathscr{B}_Z(U)) \in \mathcal{D}$. By $\mathcal{D}$-compactness, $\pi \circ f$ has a $\mathcal{D}$-limit point $A \in \mathscr{B}(Y)$. For each $p \in A$, let $D_p = (\pi \circ f)^{-1}(\mathscr{B}_Y(\{p\}))$. Then $D_p \in \mathcal{D}$ for all $p \in A$. Fix $p \in A$. Set $f_p = f \upharpoonright D_p$ an let $g_p\colon \mathscr{B}_Z(X_p) \to \mathscr{B}(X_p)$ be given by $g_p(C) = C \cap X_p$ for all $C \in \mathscr{B}_Z(X_p)$. Extend $g_p \circ f$ to a map $h_p\colon K \to \mathscr{B}(X_p)$. Let A_p be a $\mathcal{D}$-limit point of h_p. So defining A_p for all $p \in A$, set $B = \bigcup_{p\in A} A_p \in \mathscr{B}(Z)$. Suppose $U \in \mathcal{E}$ and $B \in \mathscr{B}_Z(U)$. Then there exists $p \in A$ such that $U \in \mathcal{E}_p$ and $A_p \in \mathscr{B}_{X_p}(U)$. Set $E_p = h_p^{-1}(\mathscr{B}_{X_p}(U))$. Then $E_p \in \mathcal{D}$; hence, $D_p \cap E_p \in \mathcal{D}$. But $f^{-1}(\mathscr{B}_Z(U)) = D_p \cap E_p$; hence, B is a $\mathcal{D}$-limit point of f. Thus, $\mathscr{B}(Z, \mathcal{E})$ is $\mathcal{D}$-compact. □

Theorem 5.16 is a verbatim copy of Theorem 5.1 except that "$\mathcal{D}$-compact" replaces "compact." Given this, it is easy to verify that Corollary 5.2 and Theorem 5.3 are also still hold if "$\mathcal{D}$-compact" replaces "compact": we need only copy their proofs and respectively replace "compact" and "Theorem 5.1" with "$\mathcal{D}$-compact" and "Theorem 5.16."

Our characterization of compactness of $\mathscr{B}\big(X, [X]^1\big)$ in Theorem 5.7 also has a close analogue for $\mathcal{D}$-compactness, but its proof is not as trivially translated. To simplify notation, if $\mathcal{D}$ is nonprincipal, then let δ be the maximum cardinal λ for which $\mathcal{D}$ is λ-complete. If $\mathcal{D}$ is principal, then let $\delta = \infty$, which means δ is formally greater than every ordinal.

Definition 5.17. Given $S \in \mathscr{S}(X)$, we say that S is $\mathcal{D}$-*capped* if, for every map $f\colon K \to \mathscr{B}(X)$ such that $f^{-1}(\mathscr{B}_X(\{p\})) \in \mathcal{D}$ for all $p \in S$, we have

$$f^{-1}(\mathscr{B}_X(\mathscr{F}_X(S))) \in \mathcal{D}.$$

Theorem 5.18. *The space $\mathscr{B}\big(X, [X]^1\big)$ is $\mathcal{D}$-compact if and only if every fork in X has cardinality less than δ and every proper semibranch of X is $\mathcal{D}$-capped.*

Proof. Suppose $\mathscr{B}\big(X,[X]^1\big)$ is $\mathcal{D}$-compact. Let $S\in\mathscr{S}(X)$. Let $\mathcal{A}=\bigcap_{p\in S}\mathscr{B}_X(\{p\})$. By Lemma 5.6, $\mathcal{A}$ is closed; hence, it is $\mathcal{D}$-compact. Moreover, a branch A is in $\mathcal{A}$ if and only if it contains S. Suppose $|\mathscr{F}_X(S)|\geq\delta$. Then $\delta<\infty$ and there exists a partition $\langle I_\alpha\rangle_{\alpha<\delta}$ of K such that $I_\alpha\notin\mathcal{D}$ for all $\alpha<\delta$. Let η be an injection from δ into $\mathscr{F}_X(S)$, and choose $f\colon K\to\mathcal{A}$ such that $\pi_S(f(I_\alpha))=\{\eta(\alpha)\}$. Let A be a $\mathcal{D}$-limit point of f in $\mathcal{A}$. Then $A\supseteq S$; hence, $A@S$ exists. If $A@S=\eta(\alpha)$ for some $\alpha<\delta$, then $f^{-1}(\mathscr{B}_X(\{A@S\})=I_\alpha\notin\mathcal{D}$. If $A@S$ is not in the range of η, then $f^{-1}(\mathscr{B}_X(\{A@S\})=\emptyset\notin\mathcal{D}$. Therefore, A is not a $\mathcal{D}$-limit point of f, which is absurd. Hence, $|\mathscr{F}_X(S)|<\delta$.

Let us show that S is $\mathcal{D}$-capped. Let $f\colon K\to\mathscr{B}(X)$ satisfy $f^{-1}(\mathscr{B}_X(\{p\}))$ for all $p\in S$. Let A be a $\mathcal{D}$-limit point of f. Suppose $A\not\supseteq S$. Then there exists $p\in S$ and $q\in A$ such that $p\not\leq q\not\leq p$. Hence, $\mathscr{B}_X(\{p\})\cap\mathscr{B}_X(\{q\})=\emptyset$; hence, $f^{-1}(\mathscr{B}_X(\{q\}))\notin\mathcal{D}$; hence, A is not a $\mathcal{D}$-limit point of f, which is absurd. Therefore, $A\supseteq S$. Hence, $A\in\mathscr{B}_X(\mathscr{F}_X(S))$; hence, $f^{-1}(\mathscr{B}_X(\mathscr{F}_X(S)))\in\mathcal{D}$; hence, S is $\mathcal{D}$-capped. Thus, the "only if" part of the theorem holds.

Let us prove the "if" part. Suppose every fork in X has cardinality less than δ and every proper semibranch of X is $\mathcal{D}$-capped. Let f be an arbitrary map from K to $\mathscr{B}(X)$. We recursively construct a $\mathcal{D}$-limit point A of f by defining $A@A_\alpha$ in terms of A_α for all $\alpha<h(A)$. Suppose $A_\alpha\in\mathscr{S}(X)$ and $f^{-1}(\mathscr{B}_X(\{p\}))\in\mathcal{D}$ for all $p\in A_\alpha$. Then $f^{-1}(\mathscr{B}_X(\mathscr{F}_X(A_\alpha)))\in\mathcal{D}$ because A_α is $\mathcal{D}$-capped. Since $|\mathscr{F}_X(A_\alpha)|<\delta$, there exists $q\in\mathscr{F}_X(A_\alpha)$ such that $f^{-1}(\mathscr{B}_X(\{q\}))\in\mathcal{D}$. Set $A@A_\alpha=q$. Then, by induction, $f^{-1}(\mathscr{B}_X(\{p\}))\in\mathcal{D}$ for all $p\in A$. Therefore, since $\{\mathscr{B}_X(\{p\}):p\in X\}$ is a subbasis for $\mathscr{B}\big(X,[X]^1\big)$ and $\mathcal{D}$ is a filter, A is a $\mathcal{D}$-limit point of f. □

For a tree W, the next theorem characterizes $\mathcal{D}$-compactness of $\mathscr{B}\big(W,[W]^1\big)$ similarly to Theorem 5.13. First, we need a lemma.

Lemma 5.19. *Suppose $S\in\mathscr{S}(X)$ and $|S|<\delta$. Then S is $\mathcal{D}$-capped.*

Proof. Let $f\colon K\to\mathscr{B}(X)$ satisfy $f^{-1}(\mathscr{B}_X(\{p\}))\in\mathcal{D}$ for all $p\in S$. Then we have

$$f^{-1}(\mathscr{B}_X(\mathscr{F}_X(S)))\supseteq\bigcap_{p\in S}f^{-1}(\mathscr{B}_X(\{p\}))\in\mathcal{D}$$

because $|S|<\delta$. □

Theorem 5.20. *The space $\mathscr{B}\big(W,[W]^1\big)$ is $\mathcal{D}$-compact if and only if there is a tree V such that $\mathscr{B}\big(W,[W]^1\big)$ is homeomorphic to $\mathscr{B}\big(V,[V]^1\big)$ and all forks and proper semibranches of V have cardinality less than δ.*

Proof. By Lemma 5.12, there is a tree V such that $\sim_V$ is discrete and $\mathscr{B}\big(W,[W]^1\big)$ is homeomorphic to $\mathscr{B}\big(V,[V]^1\big)$. Suppose $\mathscr{B}\big(W,[W]^1\big)$ is $\mathcal{D}$-compact. Then so is $\mathscr{B}\big(V,[V]^1\big)$. By Theorem 5.18, all forks in V have cardinality less than δ. Suppose $S\in\mathscr{S}(V)$ and $|S|\geq\delta$. Then $\delta<\infty$ and S_δ has order type δ. Let us derive a contradiction. First we show that $|\mathscr{F}_V(S_{\alpha+1})|\geq 2$ for all $\alpha<\delta$. Fix $\alpha<\delta$. Since V is a tree, $S@S_\alpha$ is the maximum strict lower bound of every element of $\mathscr{F}_V(S_{\alpha+1})$. Moreover, if $|\mathscr{F}_V(S_{\alpha+1})|=1$, then $S@S_{\alpha+1}$ is the minimum strict upper bound of $S@S_\alpha$; whence, $S@S_\alpha<' S@S_{\alpha+1}$, in contradiction with the discreteness of $\sim_V$. Therefore, $|\mathscr{F}_V(S_{\alpha+1})|\geq 2$ as desired.

By the definition of δ, there is a sequence $\langle D_\alpha\rangle_{\alpha<\delta}$ in $\mathcal{D}$ such that $\bigcap_{\alpha<\delta}D_\alpha=\emptyset$ and $D_\alpha\supseteq D_\beta$ for all $\alpha<\beta<\delta$. Moreover, we may assume that $D_\lambda=\bigcap_{\alpha<\lambda}D_\alpha$ for all limit ordinals $\lambda<\delta$. For each $\alpha<\delta$, choose $p_\alpha\in\mathscr{F}_V(S_{\alpha+1})\setminus\{S@S_{\alpha+1}\}$

and choose $A^{(\alpha)} \in \mathscr{B}_V(\{p_\alpha\})$. Since V is a tree, $S_{\alpha+1} \subseteq A_\alpha$ for all $\alpha < \delta$. Let f be the map from K to $\mathscr{B}(V)$ given by $f(D_\alpha \setminus D_{\alpha+1}) = \{A^{(\alpha)}\}$ for all $\alpha < \delta$. Then $f^{-1}(\{S@S_\alpha\}) = D_\alpha \in \mathcal{D}$ for all $\alpha < \delta$. By Theorem 5.18, S is $\mathcal{D}$-capped; hence, $f^{-1}(\mathscr{B}_V(\mathscr{F}_V(S))) \in \mathcal{D}$. But V is a tree; hence,

$$\mathscr{B}_V(\mathscr{F}_V(S)) = \{A \in \mathscr{B}(V) : A \supseteq S\};$$

hence,

$$f^{-1}(\mathscr{B}_V(\mathscr{F}_V(S))) = \bigcap_{\alpha<\delta} f^{-1}(\mathscr{B}_V(\{S@S_\alpha\})) = \bigcap_{\alpha<\delta} D_\alpha = \emptyset,$$

which is absurd. Therefore, $|S| < \delta$ as desired.

Conversely, suppose there is a tree V such that $\mathscr{B}\big(W, [W]^1\big)$ is homeomorphic to $\mathscr{B}\big(V, [V]^1\big)$ and all forks and proper semibranches of V have cardinality less than δ. By Lemma 5.19, all proper semibranches of V are $\mathcal{D}$-capped. Hence, by Theorem 5.18, $\mathscr{B}\big(V, [V]^1\big)$ is $\mathcal{D}$-compact. Hence, $\mathscr{B}\big(W, [W]^1\big)$ is $\mathcal{D}$-compact. □

Acknowledgements

This research has been partially supported by the MIT Undergraduate Research Opportunities Program (UROP), in particular, the Meryl and Stewart Robertson UROP Fund. The author especially wishes to thank his UROP supervisor Prof. Jonathan Farley.

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