Trapping Light With Mirrors

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Abstract. We show that, given finitely many line-segment mirrors in the plane, that do not touch, and an arbitrary point source of light, if all angles made by lines parallel to mirrors are rational multiples of π, then all but countably many emitted light beams escape. This result is shown to imply that, for a given point source of light, a randomly chosen configuration of finitely many nontouching line-segment mirrors has a nonzero probability of letting at least one light beam escape.

1. Introduction. In the plane, is there a configuration of finitely many line-segment mirrors, with both sides being reflective, that can trap all the light from a given point source without allowing the mirrors to touch each other? The first published mention of this problem is by O'Rourke and Petrovici [**1**], and it remains an open problem, though we develop a partial solution here. O'Rourke and Petrovici note that the same problem for circular mirrors is also open, though some other related problems have been solved; see their paper for more background on these type of problems.

To state this problem in a proper mathematical fashion, let us formulate a precise definition of light beams. The *state* of a light beam at any given moment is defined as its position and direction of travel; we assume light travels with unit speed. The initial state of a light beam determines all its subsequent states; hence, a light beam is a map from its initial state and time elapsed to its current state. The dynamics of our light beams are as in geometric optics, with light beams bouncing off of mirrors in the obvious way, without refraction, diffraction, and the like. Also, as in geometric optics, light beams do not interfere with each other.

The only tricky matter is saying what happens to a light beam that hits a mirror endpoint. Let us agree that mirror endpoints absorb all light beams. This convention might be questionable, but it is of great technical aid. Moreover, our conclusions are of the form "at least this many light beams escape"; hence, not absorbing light beams incident on mirror endpoints would not hurt our conclusions, as it would only make escape easier.

Let us make two more conventions. First, mirrors do not contain their endpoints; hence, whenever we speak of a point being on a mirror or of a light beam hitting a mirror, we are not including the possibility of that point being a mirror endpoint or of that light beam hitting a mirror endpoint. Second, if a light beam is initially on a mirror or a mirror endpoint and its initial direction is into that mirror, then the light beam is absorbed.

We also need to define the term "trapped" precisely. A light beam is said to be *trapped* if it hits mirrors infinitely many times, and hence does not get absorbed by, say, hitting a mirror endpoint. A light beam that is not trapped, nor ever absorbed, is said to *escape* because it necessarily traces out an infinitely long ray after hitting mirrors finitely many times.

Given this more precise version of the problem, we use nothing more advanced than elementary point-set topology and a tiny bit of measure theory to draw the following conclusions. For a special type of mirror configuration we can guarantee that, from each initial position, only countably many initial directions do not escape if the mirrors do not touch. This special type is the *rational* mirror configuration, where rationality means that all angles formed by lines parallel to mirrors are rational multiples of π. Moreover, for any given initial light-beam position, if we choose a random configuration of mirrors from a bounded open subset of the configuration space, then, with nonzero probability, for at least one initial direction, the light beam escapes. This last result follows from the density of the rational configuration in the configuration space, along with a continuity argument.

To reach the above conclusions, in Section 2 we set up our notation, along with two ways of modelling the dynamics of a light beam: one in which the light beam is reflected by mirrors, and one in which the mirror configuration is reflected by the light beam in a certain way. In Section 3 we prove general results applicable to all mirror configurations, including the result that, for a given initial position, only countably many initial directions produce a periodically trapped light beam. In Section 4 we build on these results, showing how rationality can reduce the three-dimensional dynamical system of a light beam's state into a one-dimensional system. Analysis of this one-dimensional system shows that in the rational case, only countably many directions of travel are attained by light beams that are aperiodically trapped. Combining this analysis with our results on periodic trapping yields our conclusions.

2. Preliminaries. We define a mirror to be a nonempty open line segment in the complex plane, in accordance with our convention that mirrors not include their endpoints. Therefore, two mirrors do not touch if and only if their closures are disjoint.

We also represent mirrors in another space. For our purposes, S^1 is defined as $\{z \in \mathbb{C} : |z| = 1\}$. Set $\mathcal{M}_1 = \mathbb{C} \times (0, \infty) \times S^1$. We then represent mirrors as elements of $\mathcal{M}_1$, with the first component of an element being the position of a mirror endpoint, the second component being the length of the mirror, and the third component being the direction in which the mirror extends from the endpoint specified by the first component. Note that mirrors are not uniquely represented: for example, $\langle 3 + 4i, 2, 1\rangle$ is the same mirror as $\langle 5 + 4i, 2, -1\rangle$.

With the map g defined below, we make explicit the correspondence between mirrors and their representations. We denote the powerset operator by $\mathcal{P}$. Define g as follows:

$$g\colon \mathcal{M}_1 \to \mathcal{P}(\mathbb{C}) \text{ by } g(x, l, u) = \{x + ru : r \in (0, l)\}\,.$$

Strictly speaking, $g(M_i)$ is the mirror represented by M_i, but it is shorter to write "the mirror M_i" when referring to $g(M_i)$. We frequently use this abbreviation.

Let n be the number of mirrors. Let the mirrors be ordered by choosing an element M in the Cartesian product $\mathcal{M}_1^n$ such that each mirror is represented by one of the coordinate projections of M. We denote the ith coordinate projection map by π_i. We abbreviate $\pi_i(M)$ with M_i, making the disjointness requirement $\overline{g(M_i)} \cap \overline{g(M_j)} = \varnothing$ for all distinct $i, j = 1, \ldots, n$.

Let $\mathcal{M}_n$ be the set of all mirror configurations in $\mathcal{M}_1^n$ such that the closures of the mirrors are pairwise disjoint. We call $\mathcal{M}_n$ the space of *legal* configurations of n mirrors.

It is easily checked that $\mathcal{M}_n$ is open in $\mathcal{M}_1^n$. Let us assume $M \in \mathcal{M}_n$.

Finally, let us define some abbreviations for our mathematical representation of mirrors. For a given legal mirror configuration H, let $\delta(H)$ denote the minimum distance between mirrors in H, and let $L(H)$ denote the maximum mirror length in H. Note that both are positive because there are finitely many mirrors, all with pairwise disjoint closures of finite length. Also, denote the projection from $\mathcal{M}_1$ to $\mathbb{C}$ by ζ, from $\mathcal{M}_1$ to $(0, \infty)$ by λ, and from $\mathcal{M}_1$ to S^1 by θ.

With the mirrors mathematically modelled, let us now represent the light beams mathematically. Take the position space to be $\mathbb{C}$ and the direction space to be S^1. Also, use angle brackets to denote ordered tuples. Given a position q_0 and a direction p_0, the ordered pair $\langle q_0, p_0 \rangle$ is a light-beam state. Suppose t is a nonnegative real. To describe the dynamics of a light beam, let $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$ denote the position and direction that a light beam attains at time t if its state at time 0 is $\langle q_0, p_0 \rangle$. If a light beam is absorbed prior to time t, then leave $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$ undefined.

Given a nonnegative integer k, let $\tau_k(q_0, p_0)$ denote the kth time at which the light beam with initial state $\langle q_0, p_0 \rangle$ hits a mirror or mirror endpoint, not counting time 0 if the light beam was initially on a mirror or mirror endpoint. If this light beam hits mirrors or mirror endpoints less than k times, then set $\tau_k(q_0, p_0) = \infty$. If $k = 0$, then set $\tau_k(q_0, p_0) = 0$. Similarly, set $\mu_k(q_0, p_0) = i$ where i is such that, at time $\tau_k(q_0, p_0)$, the light beam with initial state $\langle q_0, p_0 \rangle$ lies on the closure of the mirror M_i. If no such i exists, then set $\mu_k(q_0, p_0) = 0$. Call the sequence of indices of mirrors hit by a light beam its *hit sequence*, and say that its hit sequence is *trapped* if it contains infinitely many nonzero elements. Therefore, a light beam is trapped if and only if its hit sequence is trapped.

As described in the introduction, our light beams travel with unit speed, and between reflections, they simply travels in straight lines. Symbolically, we have the following for each nonnegative integer i:

$$q(q_0, p_0, t) = q\left(q_0, p_0, \tau_i(q_0, p_0)\right) + tp\left(u, \tau_i(q_0, p_0)\right) \text{ for } \tau_i(q_0, p_0) < t \leq \tau_{i+1}(q_0, p_0), \tag{2-1}$$

$$p(q_0, p_0, t) = p\left(q_0, p_0, \tau_i(q_0, p_0)\right) \text{ for } \tau_i(q_0, p_0) < t < \tau_{i+1}(q_0, p_0).$$

To describe what happens at reflections, we employ analytic geometry. Let us use the overline operator to represent complex conjugation as well as topological closure, using context to disambiguate. It is easily checked that

$$p\left(q_0, p_0, \tau_{i+1}(q_0, p_0)\right) = \theta\left(M_{\mu_{i+1}(q_0, p_0)}\right)^2 \overline{p\left(q_0, p_0, \tau_i(q_0, p_0)\right)}. \tag{2-2}$$

Note that the current state of a light beam determines all of its earlier states, except for light beams that are absorbed. Indeed, suppose

$$\langle q_1, p_1 \rangle = \langle q(q_0, p_0, t), p(q_0, p_0, t) \rangle.$$

If q_1 is not on a mirror or mirror endpoint, then

$$\langle q, p \rangle(q_0, p_0, s) = \langle q, -p \rangle(q_1, -p_1,\ t - s) \text{ for } s \in [0, t] - \{\tau_i(q_0, p_0) : i \geq 1\}. \tag{2-3}$$

This formula just says that to determine a light beam's past we do the obvious thing: reverse its direction and see where it goes. If q_1 lies on the mirror M_i with a direction not parallel to that mirror, then we first have to undo a reflection before reversing direction; hence, the following formula is used instead:

$$\langle q,p\rangle(q_0,p_0,s)=\langle q,-p\rangle\big(q_1,-\theta(M_i)^2\overline{p}_1,t-s\big)\text{ for }s\in[0,t]-\{\tau_i(q_0,p_0):i\geq 1\}$$

We could use this time-reversed determinism to explicitly define q and p for negative time values, but (2-3) suffices for this paper.

An alternative way to model reflections is to apply them to the complex plane instead of to the beams of light that travel across it. In this model, all light beams travel as straight lines, whereas the mirror configuration can be different for different light beams and be different for the same light beam at different times. For all light beams, the mirror configuration at time 0 is as usual. However, when a light beam hits a mirror, the mirror configuration for that light beam is reflected about the mirror hit. Thus, although all light beams start with the same mirror configuration, if they hit different mirrors or hit mirrors at different times, then they needn't have the same mirror configurations at a later time. Conversely, if, up until a certain time, two light beams have hit the same mirrors in the same order, then the light beams have the same mirror configurations at that time.

Given a nonnegative real t and a light-beam state $\langle q_0,p_0\rangle$, define $q'(q_0,p_0,t)$ and $p'(q_0,p_0,t)$ as the position and direction, respectively, that the light beam with initial state $\langle q_0,p_0\rangle$ attains at time t in this alternative model. Note that p' is constant with respect to time; hence, $q'(q_0,p_0,t)=q_0+p_0t$. Let $M'(q_0,p_0,t)$ denote the mirror configuration that that light beam has a time t. See Figure 2-1.

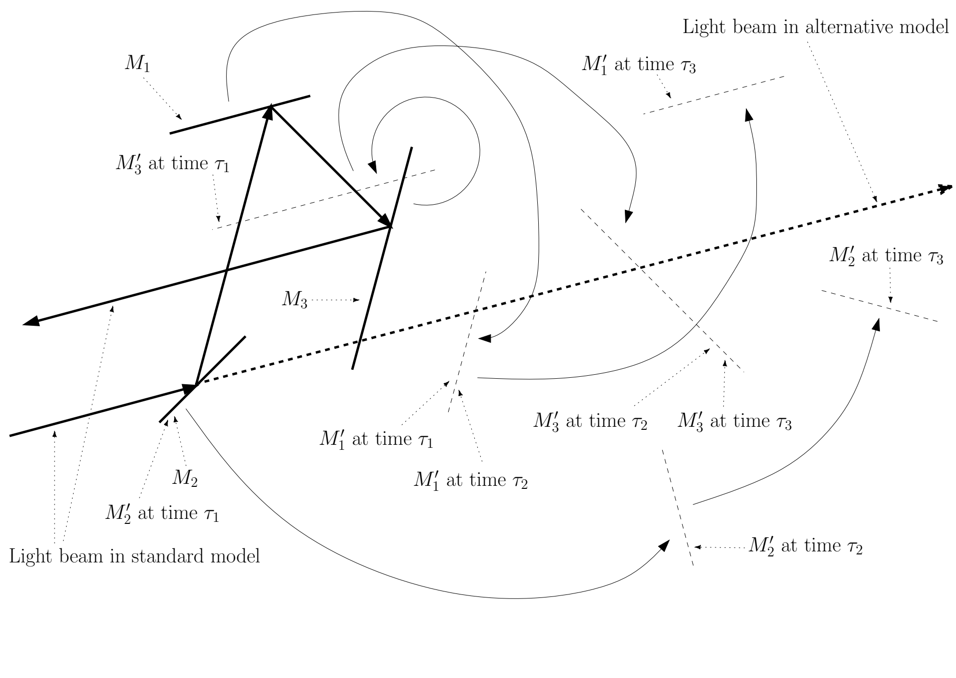

Figure 2-1. Reflecting the light beam vs. reflecting the underlying plane.

To transform $q'(q_0, p_0, t)$ to $q(q_0, p_0, t)$, we perform a finite sequence of affine transformations that depend only on the sequence of mirrors hit, before time t, by the light beam with initial state $\langle q_0, p_0 \rangle$. Specifically, on $q(q_0, p_0, t)$ we perform each reflection that was performed on the mirror configuration of the light beam up until time t, but in reverse order. It follows that, if two light beams have respective initial states $\langle x_1, v_1 \rangle$ and $\langle x_2, v_2 \rangle$, and hit the same mirrors in the same order up until respective times t_1 and t_2, then

$$|q(x_1, v_1, t_1) - q(x_2, v_2, t_2)| = |q'(x_1, v_1, t_1) - q'(x_2, v_2, t_2)| . \tag{2-4}$$

This equation is crucial to the proofs of Theorem 3-1 and Lemma 3-3.

3. General Results. In this section, we prove several theorems that apply to all legal mirror configurations. These theorems are the foundation of the next section.

Theorem 3-1. *Given two light beams with initial states $\langle x_1, v_1 \rangle$ and $\langle x_2, v_2 \rangle$, if the light beams have equal, trapped hit sequences, then $v_1 = v_2$.*

Proof: Intuitively, the two beams, represented as straight lines using q', become arbitrarily far apart after sufficient time if they are not parallel; hence, they cannot hit the same infinite sequence of mirrors. See Figure 3-1.

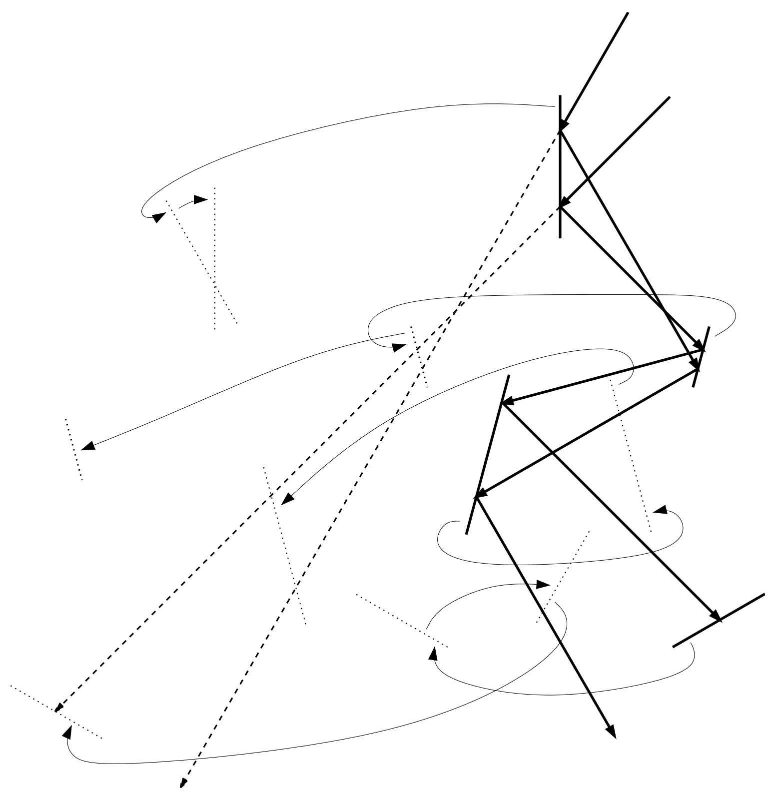

Figure 3-1. Why two light beams' hit sequences eventually disagree if they are not parallel.

Between each reflection, a light beam travels in a straight line at unit speed, and it must traverse at least the minimum distance between mirrors. Therefore, for all $k > 0$, we have $\tau_k(x_1, v_1) \geq k\delta(M)$ and $\tau_k(x_2, v_2) \geq k\delta(M)$.

Let $f(t_1, t_2) = |q'(x_1, v_1, t_1) - q'(x_2, v_2, t_2)|^2$. Then

$$f(t_1, t_2) = t_1^2 + t_2^2 - 2t_1t_2\mathrm{Re}\,(v_1\overline{v}_2) + 2t_1\mathrm{Re}\,((x_1 - x_2)\overline{v}_1) + 2t_2\mathrm{Re}\,((x_2 - x_1)\overline{v}_2) + |x_1 - x_2|^2.$$

Transforming to polar coordinates using $t_1 = r\cos(\omega)$ and $t_2 = r\sin(\omega)$, we get

$$f(t_1, t_2) = r^2\,(1 - \sin(2\omega)\mathrm{Re}(v_1\overline{v}_2)) + 2r\mathrm{Re}\,((x_1 - x_2)\overline{v}_1\cos(\omega) + (x_2 - x_1)\overline{v}_2\sin(\omega)) + |x_1 - x_2|^2.$$

Suppose $v_1 \neq v_2$. Then $\mathrm{Re}(v_1\overline{v}_2) < 1$; hence, $1 - \sin(2\omega)\mathrm{Re}(v_1\overline{v}_2)$ has a positive global minimum with respect to ω. Denote this minimum by a. Let b denote the global minimum of

$$2\mathrm{Re}\,((x_1 - x_2)\overline{v}_1\cos(\omega) - (x_1 - x_2)\overline{v}_2\sin(\omega))$$

with respect to ω. It follows that $f(t_1, t_2) \geq ar^2 + br + |x_1 - x_2|^2$. If $t_1 = \tau_k(x_1, v_1)$ and $t_2 = \tau_k(x_2, v_2)$, then $r \geq \max\{\tau_k(x_1, v_1), \tau_k(x_2, v_2)\} \geq k\delta(M)$; hence, for sufficiently large k, we have

$$f\,(\tau_k(x_1, v_1), \tau_k(x_2, v_2)) > L(M)^2. \tag{3-1}$$

The light beams hit the same sequence of mirrors; hence, by (2-4), the distance

$$\left|q\,(x_1, v_1, \tau_k(x_1, v_1)) - q\,(x_2, v_2, \tau_k(x_2, v_2))\right| \tag{3-2}$$

is unchanged if q' replaces q; hence, (3-1) implies this distance gets arbitrarily large as k increases. However, at times $\tau_k(x_1, v_1)$ and $\tau_k(x_2, v_2)$, the light beams with respective initial states $\langle x_1, v_1\rangle$ and $\langle x_2, v_2\rangle$ hit the same mirror; consequently, the distance between their positions, which is exactly the quantity in (3-2), is not greater than the length of that mirror, and hence not greater than $L(M)$. We therefore conclude that $v_1 = v_2$. □

We say that a direction p_0 is *aperiodic* if there is no complex number q_0 such that the light beam with initial state $\langle q_0, p_0\rangle$ has a periodic trapped hit sequence.

Corollary 3-2. *All but countably many directions are aperiodic.*

Proof: There are only countably many periodic trapped hit sequences because the set of all periodic infinite sequences of elements of $\{1, \ldots, n\}$ is countable. By Theorem 3-1, there is at most one initial direction for which a light beam attains a given trapped hit sequence, regardless of initial position. □

We note that O'Rourke and Petrovici provide a different proof of Corollary 3-2 in [**1**, p. 138].

The proof of Theorem 3-1 very similar to the proof of the following lemma. This lemma is used to prove Theorem 3-4.

Lemma 3-3. *Suppose that two light beams have initial states $\langle q_0 + s_1p_1, p_1\rangle$ and $\langle q_0 + s_2p_2, p_2\rangle$ where $q_0 \in \mathbb{C}$, where $s_1, s_2 \in [0, \infty)$, where $p_1, p_2 \in S^1$, and where $p_1 \neq p_2$. Also suppose that there is a positive integer k such that the first k elements of their hit sequences agree and each light beam is not absorbed prior to time τ_{k+1} evaluated at that light beam's initial state. For each light beam, consider the line segment formed* *by the light beam's positions from times τ_k to τ_{k+1}. Then the distance between these two line segments is positive:*

$$0 < \inf\big\{|q(q_0 + s_1 p_1, p_1, t_1) - q(q_0 + s_2 p_2, p_2, t_2)| : \\ t_i \in [\tau_k(q_0 + s_i p_i, p_i), \tau_{k+1}(q_0 + s_i p_i, p_i)) \text{ for all } i = 1, 2\big\}.$$

Proof: Figure 3-2 makes the lemma obvious; we just need to express this figure algebraically.

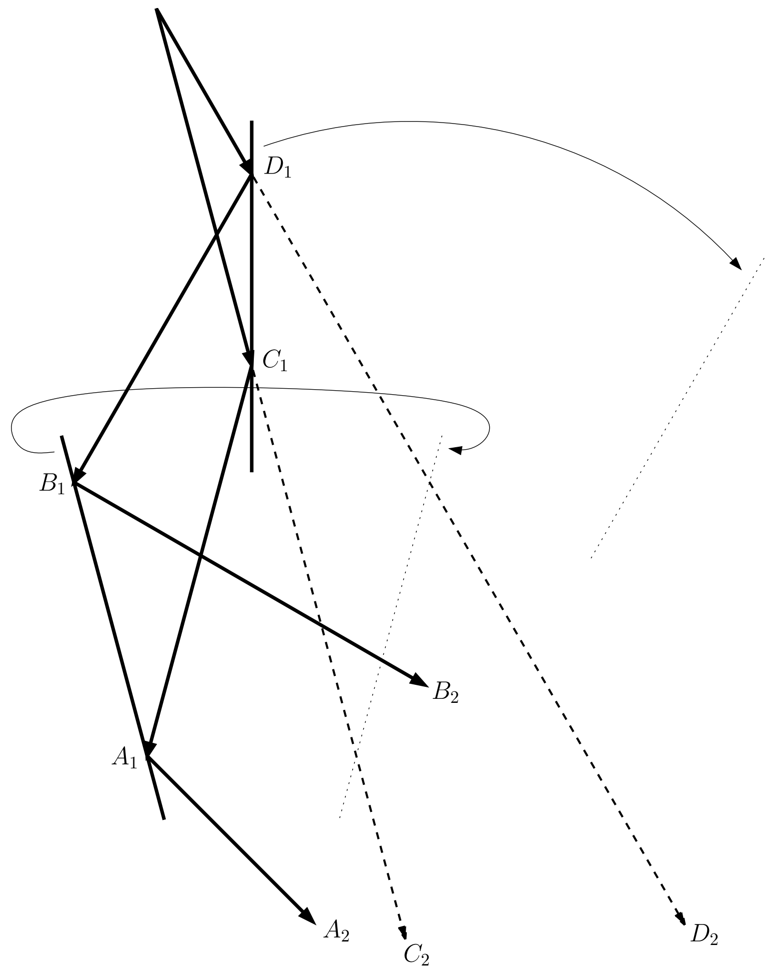

Figure 3-2. Rays $\overrightarrow{C_1C_2}$ and $\overrightarrow{D_1D_2}$ cannot intersect; hence, Rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{B_1B_2}$ cannot intersect.

Set $q_1 = q_0 + s_1 p_1$ and $q_2 = q_0 + s_2 p_2$. For all t_1 in $[\tau_k(q_1, p_1),\ \tau_{k+1}(q_1, p_1))$ and for all t_2 in $[\tau_k(q_2, p_2),\ \tau_{k+1}(q_2, p_2))$, the sequence of mirrors hit before time t_1 by the light beam with initial state $\langle q_1, p_1 \rangle$ agrees with the sequence of mirrors hit before time t_2

by the light beam with initial state $\langle q_2, p_2\rangle$; hence, we have

$$
\begin{aligned}
|q(q_1,p_1,t_1) - q(q_2,p_2,t_2)| &= |q'(q_1,p_1,t_1) - q'(q_2,p_2,t_2)| \\
&= |q_0 + p_1(s_1+t_1) - (q_0 + p_2(s_2+t_2))| \\
&= \sqrt{(s_1+t_1)^2 - 2(s_1+t_1)(s_2+t_2)\mathrm{Re}(p_1\overline{p}_2) + (s_2+t_2)^2}.
\end{aligned}
$$

Transforming to polar coordinates using $s_1 + t_1 = r\cos(\omega)$ and $s_2 + t_2 = r\sin(\omega)$, we obtain

$$
\begin{aligned}
|q(q_1,p_1,t_1) - q(q_2,p_2,t_2)| &= r\sqrt{1 - \mathrm{Re}(p_1\overline{p}_2)\sin(2\omega)} \\
&\geq \max\{s_1+t_1, s_2+t_2\}\sqrt{1 - \mathrm{Re}(p_1\overline{p}_2)\sin(2\omega)}.
\end{aligned}
$$

Since $p_1 \neq p_2$, we have $\mathrm{Re}(p_1\overline{p}_2) < 1$; hence, $\sqrt{1 - \mathrm{Re}(p_1\overline{p}_2)\sin(2\omega)}$ has a positive global minimum with respect to ω. Let A denote this minimum. It follows that

$$
\begin{aligned}
\inf\bigl\{|q(q_1,p_1,t_1) - q(q_2,p_2,t_2)| : t_i \in [\tau_k(q_i,p_i), \tau_{k+1}(q_i,p_i)) \text{ for all } i = 1,2\bigr\} \\
\geq \max\{s_1 + \tau_k(q_1,p_1), s_2 + \tau_k(q_2,p_2)\}A.
\end{aligned}
$$

Since k is positive, $\tau_k(q_1,p_1)$ and $\tau_k(q_2,p_2)$ are both positive; hence, the right side of the above inequality is positive. □

Before stating Theorem 3-4, we need to make some definitions. Given any two, not necessarily distinct, complex numbers x and x', we define a *path* from x to x' to be a light-beam state $\langle q_0, p_0\rangle$ where q_0 is not on a mirror or a mirror endpoint, and where there exist positive reals t and t' such that $q(q_0, -p_0, t) = x$ and $q(q_0, p_0, t') = x'$. If u is a direction, then we say u *allows* a path from x to x' if there exists a complex number y such that $\langle y, u\rangle$ is a path from x to x'. We say that a direction is *degenerate* if it allows a path from a mirror endpoint to a mirror endpoint.

Theorem 3-4. *For any two complex numbers x and x', there are only countably many directions that allow a path from x to x'.*

Proof: Suppose $\langle q_0, p_0\rangle$ is a path from x to x'. We first show that this hypothesis implies the existence of a path from x to x' with a position component arbitrarily close to x. Let t satisfy $q(q_0, -p_0, t) = x$, and let t' satisfy $q(q_0, p_0, t') = x'$. The light beam with initial state $\langle q_0, -p_0\rangle$ hits mirrors finitely many times before reaching x; hence, it travels at constant direction from the last of these hit points to x. Let u be the opposite of this direction. Let ϵ be a positive real less than the infimum of distances from x to any point on the closure of any mirror, except for the closure of the mirror x is on, if it is indeed on the closure of some mirror. Then $x + \epsilon u$ is not on a mirror or a mirror endpoint, and there exists an s in $(0, t)$ such that $q(q_0, -p_0, s) = x + \epsilon u$. It follows from (2-3) that $\langle q, p\rangle(x + \epsilon u, u, s) = \langle q_0, p_0\rangle$; hence, $q(x + \epsilon u, u, s + t) = x'$. Also, $q(x + \epsilon u, -u, \epsilon) = x$; hence, $\langle x + \epsilon u, u\rangle$ is a path from x to x'.

We next show that there are only countably many directions w such that $\langle x + \epsilon w, w\rangle$ is a path from x to x'. Suppose that v is in $S^1 - \{u\}$, and $\langle x + \epsilon v, v\rangle$ is a path from x to x'. Then the light beams with initial states $\langle x + \epsilon u, u\rangle$ and $\langle x + \epsilon v, v\rangle$ both hit mirrors finitely many times before reaching x'. Moreover, at least one of these light beams must hit at least one mirror; otherwise, the light beams travel in straight lines forever, making it impossible for both to reach x'.

Suppose that both beams hit the same finite sequence of mirrors before reaching x', and denote the sequence's length by k. Then Lemma 3-3 implies that

$$0 < \inf\bigl\{|q(x+\epsilon u, u, t_1) - q(x+\epsilon v, v, t_2)| : t_1 \in [\tau_k(x+\epsilon u, u), \tau_{k+1}(x+\epsilon u, u)) \text{ and } t_2 \in [\tau_k(x+\epsilon v, v), \tau_{k+1}(x+\epsilon v, v))\bigr\}.$$

However, both light beams reach x' at, or prior to, the times of their $(k+1)$th mirror hits, their escape, or their absorption; hence, the above infimum must be zero, a contradiction. Therefore, u and v must correspond to different finite sequences of mirrors. Therefore, there is an injection from the set of all directions w such that $\langle x+\epsilon w, w\rangle$ is a path from x to x' to the countable set of finite sequences of elements of $\{1,\dots,n\}$; hence, the former set is countable too.

Furthermore, for each w in the former set, the light beam with initial state $\langle x+\epsilon w, w\rangle$ attains only finitely many directions before hitting x', and the light beam with initial state $\langle x+\epsilon w, -w\rangle$ attains only one direction before hitting x. Therefore, w corresponds to finitely many directions that allow a path from x to x'; hence, the set of all directions allowing paths from x to x' is countable. □

Another useful result is the equation of periodic hit sequences and periodic light-beam states. It allows us to speak of periodically and aperiodically trapped light beams without any confusion. We defer the proof to the next section.

Theorem 3-5. *Suppose the light beam with initial state $\langle x_0, v_0\rangle$ has a trapped periodic hit sequence; i.e., there is a positive integer P such that*

$$\langle \mu_k(x_0, v_0)\rangle_{k=1}^{\infty} = \langle \mu_{k+P}(x_0, v_0)\rangle_{k=1}^{\infty}.$$

Then the light beam's state is also periodic.

Up to this point, all the properties we have proved about light beams have been geometrical. These properties, though useful, are not powerful enough for the analysis of light-beam dynamics we perform in the next section. Some topological results are needed. Essentially, we need to show that the current state of a light beam is continuous with respect to both its initial state and the mirror configuration, though we phrase our topological results in terms of open sets rather than continuity.

The first step in this task is a lemma, which gives an explicit recursive relationship between a light beam's states at the times of mirror hits. Expressing this relationship succinctly requires some abbreviations. Let $\langle q_0, p_0\rangle$ be a light-beam state. For all positive integers i, set

$$q_i = q(q_0, p_0, \tau_i(q_0, p_0)), \quad p_i = p(q_0, p_0, \tau_i(q_0, p_0)), \tag{3-3}$$

$$\zeta_i = \zeta(M_{\mu_i(q_0,p_0)}), \quad \lambda_i = \lambda(M_{\mu_i(q_0,p_0)}), \quad \theta_i = \theta(M_{\mu_i(q_0,p_0)}). \tag{3-4}$$

Lemma 3-6. *For all positive integers i such that the light beam with initial state $\langle q_0, p_0\rangle$ hits mirrors at least i times, we have*

$$q_i = \frac{\zeta_i + \theta_i^2\overline{\zeta}_i + \theta_i^2(\overline{q}_{i-1} - \overline{p}_{i-1}^2 q_{i-1})}{1 - \theta_i^2\overline{p}_{i-1}^2} \text{ and } p_i = \theta_i^2\overline{p}_{i-1}. \tag{3-5}$$

Proof: The second equation follows immediately from (2-2). Let us prove the first equation. First, notice that (2-1) implies that there exists a real number t such that $q_i = q_{i-1} + tp_{i-1}$. Likewise, since $q_i \in g(M_{\mu_i(q_0,\, p_0)})$, there exists a real number l such that $q_i = \zeta_i + l\theta_i$. Therefore, $\overline{\theta}_i(q_i - \zeta_i)$ and $\overline{p}_{i-1}(q_i - q_{i-1})$ are both real. Setting these expressions equal to their conjugates gives two linear equations for q_i and $\overline{q}_i$:

$$\overline{\theta}_i(q_i - \zeta_i) = \theta_i(\overline{q}_i - \overline{\zeta}_i), \text{ and } \overline{p}_{i-1}(q_i - q_{i-1}) = p_{i-1}(\overline{q}_i - \overline{q}_{i-1}).$$

Solving the second equation for $\overline{q}_i$ and substituting into the first equation, we find that

$$(1 - \theta_i^2\overline{p}_{i-1}^2)q_i = \zeta_i + \theta_i^2\overline{\zeta}_i + \theta_i^2\overline{q}_{i-1} - \theta_i^2\overline{p}_{i-1}^2 q_{i-1}.$$

The lemma follows at once, assuming $\theta_i^2\overline{p}_{i-1}^2 \neq 1$. Suppose this assumption fails. Then $p_{i-1} = \pm\theta_i$; hence, the light beam's direction is parallel to the mirror it hits when it hits it, which is impossible, for mirrors do not include their endpoints. □

Theorem 3-7. *Given an initial light-beam state, the set of legal configurations of n mirrors that allow the light beam to escape is an open subset of the set of all legal configurations; i.e., it is open in $\mathcal{M}_n$. Moreover, given a finite sequence of mirror indices σ, let $\mathcal{W}$ denote let the set of legal configurations of n mirrors for which σ is initial segment of the light beam's hit sequence. Then $\mathcal{W}$ is also open.*

Proof: We first prove that, if the light beam escapes for the mirror configuration M, then there exists a neighborhood of M in $\mathcal{M}_n$, such that, for every element of this neighborhood, the light beam still escapes. Along the way, we show that, if σ is an initial segment of the light beam's hit sequence for the mirror configuration M, then there exists a neighborhood of M in $\mathcal{M}_n$, such that, for every element of this neighborhood, σ is still an initial segment of the light beam's hit sequence.

First we define some notation. Let the light beam have initial state $\langle q_0, p_0\rangle$. We may assume $\langle q_0, p_0\rangle$ does not lie on the closure of a mirror, for, if it does, then we can replace $\langle q_0, p_0\rangle$ with $\langle q_0 + p_0\delta(M)/2, p_0\rangle$ without changing the light beam's hit sequence. If the light beam escapes for M, let N be the number of times the light beam hits mirrors. Otherwise, let N be the length of σ.

Given $H \in \mathcal{M}_n$, set

$$D_H = \big\langle \zeta(\pi_1(H)), \theta(\pi_1(H)), \lambda(\pi_1(H)), \overline{\zeta(\pi_1(H))}, \overline{\theta(\pi_1(H))}, \overline{\lambda(\pi_1(H))}, \ldots, \\ \zeta(\pi_n(H)), \theta(\pi_n(H)), \lambda(\pi_n(H)), \overline{\zeta(\pi_n(H))}, \overline{\theta(\pi_n(H))}, \overline{\lambda(\pi_n(H))}\big\rangle.$$

Let i be an arbitrary element of $\{1, \ldots, N\}$. By induction on Lemma 3-6, q_i and p_i are rational functions of q_0, p_0, and D_M. Throughout this proof, we fix q_0 and p_0. Therefore, we may think of q_i and p_i, as well as ζ_i, θ_i, and λ_i, as rational functions on $\mathbb{C}^{6n}$ evaluated at D_M. Let $\boldsymbol{q}_i$, $\boldsymbol{p}_i$, $\boldsymbol{\zeta}_i$, $\boldsymbol{\theta}_i$, and $\boldsymbol{\lambda}_i$ be the corresponding rational functions.

For each $i = 1, \ldots, N$, define P_i, Z_i, Θ_i, and Λ_i as follows:

$$\begin{aligned}
&P_i\colon \mathcal{M}_n \to S^1 \text{ by } P_i(H) = \boldsymbol{p}_i(D_H),\\
&Z_i\colon \mathcal{M}_n \to \mathbb{C} \text{ by } Z_i(H) = \boldsymbol{\zeta}_i(D_H),\\
&\Theta_i\colon \mathcal{M}_n \to S^1 \text{ by } \Theta_i(H) = \boldsymbol{\theta}_i(D_H),\\
&\Lambda_i\colon \mathcal{M}_n \to (0,\infty) \text{ by } \Lambda_i(H) = \boldsymbol{\lambda}_i(D_H).
\end{aligned}$$

Each of these functions is continuous: Z_i, Θ_i, and Λ_i are just projections, and, by induction using Lemma 3-6, P_i is a polynomial function of $\Theta_1,\dots,\Theta_N$, $\overline{\Theta}_1,\dots,\overline{\Theta}_N$.

Before we can similarly define $Q_1,\dots,Q_N$, we must find a way to avoid division by zero in (3-5). This problem occurs if and only if $\Theta_i^2(H) = P_{i-1}^2(H)$ for some $i \le N$. Let us define $\mathcal{M}'$ as $\mathcal{M}_n$ with all these problematic mirror configurations removed:

$$\mathcal{M}' = \{H \in \mathcal{M}_n : \Theta_i^2(H) \neq P_{i-1}^2(H) \text{ for } i = 1,\dots,N\}. \tag{3-6}$$

Now, Θ_i and P_i are continuous for $i = 1,\dots,N$; hence, $\mathcal{M}'$ is open in $\mathcal{M}_n$. Also, note that $M \in \mathcal{M}'$ for the following reason: since the light beam hits mirrors N times for M, we have $\theta_i^2 \neq p_{i-1}^2$ for $i = 1,\dots,N$.

Finally, we define $Q_1,\dots,Q_N$ as follows:

$$Q_i\colon \mathcal{M}' \to \mathbb{C} \text{ by } Q_i(H) = \boldsymbol{q}_i(D_H).$$

By induction, each Q_i is a rational function of P_1, Z_1, $\Theta_1,\dots,P_{i-1}$, Z_{i-1}, Θ_{i-1}, and their complex conjugates. Also, the domain of each of $Q_1,\dots,Q_N$ was specifically chosen to avoid division by zero. Each of these functions is therefore continuous.

Let us construct a neighborhood of M that preserves the first N elements of the light beam's hit sequence. For $i = 1,\dots,N$, let $\alpha_i(M)$ be the unique s in $(0,1)$ such that $q_i = \zeta_i + s\lambda_i\theta_i$. Then $\alpha_i(M) = \overline{\theta}_i(q_i - \zeta_i)/\lambda_i$; hence, $\alpha_i(M)$ is the rational function $\overline{\boldsymbol{\theta}}_i(\boldsymbol{q}_i - \boldsymbol{\zeta}_i)/\boldsymbol{\lambda}_i$ on $\mathbb{C}^{6n}$ evaluated at D_M. For each H in $\mathcal{M}'$, set

$$\alpha_i(H) = (\overline{\boldsymbol{\theta}}_i(\boldsymbol{q}_i - \boldsymbol{\zeta}_i)/\boldsymbol{\lambda}_i)(D_H).$$

Clearly α_i is continuous on $\mathcal{M}'$. Moreover, α_i is always real: by Lemma 3-6, we have

$$\begin{aligned}
\frac{\overline{\boldsymbol{\theta}}_i}{\boldsymbol{\lambda}_i}(\boldsymbol{q}_i - \boldsymbol{\zeta}_i) &= \frac{\boldsymbol{\zeta}_i + \boldsymbol{\theta}_i^2\overline{\boldsymbol{\zeta}}_i + \boldsymbol{\theta}_i^2(\overline{\boldsymbol{q}}_{i-1} - \overline{\boldsymbol{p}}_{i-1}^2\boldsymbol{q}_{i-1}) - \boldsymbol{\zeta}_i(1 - \boldsymbol{\theta}_i^2\overline{\boldsymbol{p}}_{i-1}^2)}{\boldsymbol{\lambda}_i\boldsymbol{\theta}_i(1 - \boldsymbol{\theta}_i^2\overline{\boldsymbol{p}}_{i-1}^2)} \\
&= \frac{\boldsymbol{\zeta}_i\overline{\boldsymbol{\theta}}_i\boldsymbol{p}_{i-1} + \overline{\boldsymbol{\zeta}}_i\boldsymbol{\theta}_i\boldsymbol{p}_{i-1} + \boldsymbol{\theta}_i(\overline{\boldsymbol{q}}_{i-1}\boldsymbol{p}_{i-1} - \overline{\boldsymbol{p}}_{i-1}\boldsymbol{q}_{i-1}) - \boldsymbol{\zeta}_i(\overline{\boldsymbol{\theta}}_i\boldsymbol{p}_{i-1} - \boldsymbol{\theta}_i\overline{\boldsymbol{p}}_{i-1})}{\boldsymbol{\lambda}_i\boldsymbol{\theta}_i(\overline{\boldsymbol{\theta}}_i\boldsymbol{p}_{i-1} - \boldsymbol{\theta}_i\overline{\boldsymbol{p}}_{i-1})} \\
&= \frac{\operatorname{Re}(\overline{\boldsymbol{\zeta}}_i\boldsymbol{p}_{i-1}) + \operatorname{Im}(\overline{\boldsymbol{q}}_{i-1}\boldsymbol{p}_{i-1})}{\boldsymbol{\lambda}_i\operatorname{Im}(\overline{\boldsymbol{\theta}}_i\boldsymbol{p}_{i-1})}.
\end{aligned}$$

In order to ensure that the first N hits are preserved by a neighborhood of M, we must ensure that, for $i = 1,\dots,N$ and for all H in our neighborhood, $Q_i(H)$ lies on the mirror $\pi_{\mu_i(q_0,p_0)}(H)$. Hence, we require that our neighborhood be contained in $\bigcap_{i=1}^N \alpha_i^{-1}(0,1)$. This set is a neighborhood of M because $\alpha_i(M) \in (0,1)$ and α_i is continuous for $i = 1,\dots,N$.

We also must ensure, for all H in our neighborhood and for $i = 1,\dots,N$, that no mirrors or their endpoints get in the way as the light beam travels in a straight line from $Q_{i-1}(H)$ to $Q_i(H)$. Defined below, for each $i = 1,\dots,N$, is $\beta_i(H)$, the minimum distance from this straight line to any mirrors other than the ones at $Q_{i-1}(H)$ and $Q_i(H)$:

$$\beta_{i,j}\colon \mathcal{M}' \times \mathbb{R} \times \mathbb{R} \to [0,\infty) \text{ and } \beta_i\colon \mathcal{M}' \to [0,\infty) \text{ given by}$$

$$\begin{aligned}
\beta_{i,j}(H,x,y) = \Big| Q_{i-1}(H) &+ x|Q_i(H) - Q_{i-1}(H)|P_{i-1}(H) \\
&- \zeta(\pi_j(H)) - y\lambda(\pi_j(H))\theta(\pi_j(H))\Big|,
\end{aligned}$$

$$\beta_i(H) = \inf \bigcup_j \beta_{i,j}(\{H\} \times (0,1) \times (0,1))$$

where the union is taken over all j in $\{1, \ldots, n\} - \{\mu_{i-1}(q_0, p_0), \mu_i(q_0, p_0)\}$. We require that, for $i = 1, \ldots, N$ and for all H in our neighborhood, we have $\beta_i(H) \neq 0$; hence, we require that our neighborhood be contained in $\bigcap_{i=1}^N \beta_i^{-1}(0, \infty)$. By hypothesis, we have $\beta_i(M) > 0$; hence, $\bigcap_{i=1}^N \beta_i^{-1}(0, \infty)$ is a neighborhood of M, provided β_i is continuous for $i = 1, \ldots, N$.

Let us show each β_i is indeed continuous. Let H be an arbitrary element of $\mathcal{M}'$. Because $\beta_i(H)$ is the infimum of a finite union of sets, it is also the minimum of the infimums of these sets. Since the minimum of finitely many continuous functions is also continuous, we only need to prove that

$$\inf \beta_{i,j}\big(\{H\} \times (0,1) \times (0,1)\big) \tag{3-7}$$

is continuous with respect to H for $j = 1, \ldots, n$, except $\mu_{i-1}(q_0, p_0)$ and $\mu_i(q_0, p_0)$.

About every H in $\mathcal{M}'$, there is a neighborhood $\mathcal{U}$ and a compact set $\mathcal{C}$ such that $H \in \mathcal{U} \subseteq \mathcal{C} \subseteq \mathcal{M}'$. Since $\beta_{i,j}$ is clearly continuous on $\mathcal{M}' \times \mathbb{R} \times \mathbb{R}$, it is uniformly continuous on the compact subset $\mathcal{C} \times [0,1] \times [0,1]$. For every $\langle x, y\rangle$ in $(0,1) \times (0,1)$, let $\beta_{i,j,x,y}$ be the map from $\mathcal{M}'$ to $[0, \infty)$ given by restricting $\beta_{i,j}$ to $\mathcal{M}' \times \{x\} \times \{y\}$. Denote this family of maps by $\mathcal{B}_{i,j}$. The uniform continuity of $\beta_{i,j}$ on $\mathcal{U} \times (0,1) \times (0,1)$ implies equicontinuity of $\mathcal{B}_{i,j}$ on $\mathcal{U}$. Hence, the infimum of $\mathcal{B}_{i,j}$ is continuous on $\mathcal{U}$. Since H is an arbitrary element of $\mathcal{M}'$, this infimum is continuous on all of $\mathcal{M}'$. But this infimum is exactly the expression in (3-7).

Given this continuity, $\bigcap_{i=1}^N (\alpha_i^{-1}(0,1) \cap \beta_i^{-1}(0, \infty))$ is a neighborhood of M that preserves the the first N elements of the hit sequence of the light beam, proving the second half of the theorem.

Suppose the light beam escapes for M. Since the light beam escapes after hitting mirrors N times, the set of subsequenct positions attained by the light beam is a ray that does not intersect the closure of any mirror. For a sufficiently small neighborhood $\mathcal{V}$ of M, we may choose a positive real R such that, for any H in $\mathcal{V}$, an open disk in $\mathbb{C}$ of radius R contains all the closures of mirrors of H. Given $H \in \mathcal{M}'$, let $\gamma(H)$ be the minimum distance from all mirrors, except the mirror $\pi_{\mu_N(q_0,p_0)}(H)$, to the ray extended from $Q_N(H)$ in the direction $P_N(H)$ for length $2R$:

$$\gamma_i \colon \mathcal{M}' \times \mathbb{R} \times \mathbb{R} \to [0, \infty) \text{ and } \gamma \colon \mathcal{M}' \to [0, \infty) \text{ given by}$$

$$\gamma_i(H, x, y) = |Q_N(H) + xP_N(H) - \zeta(\pi_i(H)) - y\lambda(\pi_i(H))\theta(\pi_i(H))|,$$

$$\gamma(H) = \inf \bigcup_i \gamma_i(\{H\} \times (0, 2R) \times (0,1)),$$

where the union is taken over all i in $\{1, \ldots, n\} - \{\mu_N(q_0, p_0)\}$. We require that all H in our neighborhood satisfy $\gamma(H) \neq 0$; hence, we require that our neighborhood be contained in $\gamma^{-1}(0, \infty)$. By hypothesis, $\gamma(M) > 0$; hence, $\gamma^{-1}(0, \infty)$ is a neighborhood of M if γ is continuous. Indeed γ is continuous; the proof is the same as the proof of continuity of each of $\beta_1, \ldots, \beta_N$.

Therefore,

$$\bigcap_{1 \le i \le N} \big(\alpha_i^{-1}(0,1) \cap \beta_i^{-1}(0, \infty) \cap \gamma^{-1}(0, \infty) \cap \mathcal{V}\big)$$

is a neighborhood of M such that, for each of its elements, the light beam hits the same sequence of mirrors as it does for M and then escapes. □

Corollary 3-8. *Consider the set of legal configurations of n mirrors that allow at least one light beam emitted from a given initial position to escape. This set is open.*

Proof: Given the initial position, for each initial direction v in S^1, let $\mathcal{U}_v$ be the set of legal configurations of n mirrors that allow the light beam with the given initial position and initial direction v to escape. Then $\mathcal{U}_v$ is open for all v in S^1; hence, $\bigcup_{v \in S^1} \mathcal{U}_v$ is open. □

Theorem 3-7 asserts that each finite initial segment of a light beam's hit sequence is invariant under sufficiently small changes in the mirror configuration, and that the property of escaping is also so invariant. This assertion implies the same invariances with respect to sufficiently small changes in a light beam's initial state. The implication is through the relation between changing the initial of a light-beam state and changing the mirror configuration. Changing the initial direction is the same as rotating the mirror configuration; changing the initial position is the same as translating the mirror configuration.

Theorem 3-9. *Let N be a nonnegative integer and let $\langle q_0, p_0 \rangle$ be an initial light-beam state. Suppose that q_0 is not on any mirror or mirror endpoint, and that the light beam with initial state $\langle q_0, p_0 \rangle$ hits mirrors at least N times before escaping of being absorbed. Then there is a neighborhood U of $\langle q_0, p_0 \rangle$ in $\mathbb{C} \times S^1$ such that, for every $\langle q_0', p_0' \rangle$ in U, the light beams with initial states $\langle q_0, p_0 \rangle$ and $\langle q_0', p_0' \rangle$ agree on the first N elements of their hit sequences. Also, let W denote the set of all light-beam states $\langle q_0', p_0' \rangle$ such that $\langle q_0', p_0' \rangle$ is not on the closure of a mirror, and the light beam with initial state $\langle q_0', p_0' \rangle$ escapes after hitting mirrors exactly N times. Then W is open.*

Proof: Define a continuous map f from $\mathbb{C} \times S^1$ to $\mathcal{M}_1^n$ by the following equations, in which $i = 1, \ldots, n$ and $x \in \mathbb{C}$ and $u \in S^1$ are arbitrary:

$$\begin{aligned}
\zeta(\pi_i(f(x,u))) &= (\zeta(M_i) + q_0 - x) p_0 \overline{u}, \\
\lambda(\pi_i(f(x,u))) &= \lambda(M_i), \\
\theta(\pi_i(f(x,u))) &= \theta(M_i) p_0 \overline{u}.
\end{aligned}$$

Then, from elementary geometry, the light beam with initial state $\langle q_0', p_0' \rangle$ has the same hit sequence for mirror configuration $f(q_0', p_0')$ as the light beam with initial state $\langle q_0, p_0 \rangle$ has for mirror configuration M. Moreover, $f(q_0', p_0')$ is contained in the space of legal mirror configurations $\mathcal{M}_n$, for f preserves δ. Therefore, the light beams with initial states $\langle q_0, p_0 \rangle$ and $\langle q_0', p_0' \rangle$ agree on the first N elements of their hit sequences if and only if the hit sequence of light beam with initial state $\langle q_0, p_0 \rangle$ has the same initial segment of length N for M as it does for $f(q_0', p_0')$.

Let $\mathcal{U}$ be a neighborhood of M such that $\mathcal{U} \subseteq \mathcal{M}_n$, and, for every element H in $\mathcal{U}$, the first N elements of the hit sequence of the light beam with initial state $\langle q_0, p_0 \rangle$ is the same for M and H. Set $U = f^{-1}(\mathcal{U})$. Then U has the properties stated in the theorem. The proof of the second half of the theorem is analogous. □

There is an analogue to Theorem 3-9 for light beams with initial states lying on a mirror. When we only consider light-beam positions that lie on a particular mirror, let us use the word "offset." Given $1 \leq i \leq n$ and $x \in (0, \lambda(M_i))$, if we are only considering light-beam states lying on the mirror M_i, then x is the *offset* corresponding to the position $\zeta(M_i) + x\theta(M_i)$.

Corollary 3-10. *Let N and i be nonnegative integers such that $1 \leq i \leq n$. Let p_0 be in S^1 and x_0 be in $(0, \lambda(M_i))$. Suppose that the light beam with initial state $\langle \zeta(M_i) + x_0\theta(M_i), p_0\rangle$ hits mirrors at least N times before being absorbed or escaping. Then there is a neighborhood U of $\langle x_0, p_0\rangle$ in $(0, \lambda(M_i)) \times S^1$, such that, for every $\langle x_0', p_0'\rangle$ in U, the light beams with initial states $\langle \zeta(M_i) + x_0'\theta(M_i), p_0'\rangle$ and $\langle \zeta(M_i) + x_0\theta(M_i), p_0\rangle$ agree on the first N elements of their hit sequences. Also, let W denote the set of offsets and directions $\langle x_0', p_0'\rangle$ in $(0, \lambda(M_i)) \times S^1$ for which the light beam with initial state $\langle \zeta(M_i) + x_0'\theta(M_i), p_0'\rangle$ escapes after hitting mirrors exactly N times. Then W is open.*

Proof: We prove the first assertion of the corollary; the proof of the second is analogous. Let $f(x) = \zeta(M_i) + x\theta(M_i)$ for all real x. Then, from the definition of $\delta(M)$, it follows that, for all $\langle x_0', p_0'\rangle$ in $(0, \lambda(M_i)) \times S^1$, the light beams with initial states $\langle f(x_0'), p_0'\rangle$ and $\langle f(x_0') + p_0'\delta(M)/2, p_0'\rangle$ have the same hit sequence, except when p_0' is parallel to the mirror M_i. Let V be a neighborhood of $\langle f(x_0) + p_0\delta(M)/2, p_0\rangle$ in $\mathbb{C} \times S^1$ such that the first N elements of the hit sequence of any light beam with initial state in V agrees with the first N elements of the hit sequence of the light beam with initial state $\langle f(x_0) + p_0\delta(M)/2, p_0\rangle$. Since $N = 0$ if $p_0 = \pm\theta(M_i)$, we may assume V does not contain the closed set $\mathbb{C} \times \{\pm\theta(M_i)\}$. Let $h(x, u) = \langle f(x) + u\delta(M)/2, u\rangle$ for all $\langle x, u\rangle$ in $\mathbb{R} \times S^1$. Then h is a continuous map of $\mathbb{R} \times S^1$ into $\mathbb{C} \times S^1$. Therefore, take U to be $\big((0, \lambda(M_i)) \times S^1\big) \cap h^{-1}(V)$. □

4. Rational Mirror Configurations. Let $\mathcal{Q}_n$ denote the set of all representations of rational mirror configurations in $\mathcal{M}_n$, recalling that a mirror configuration is rational if and only if all angles made by lines parallel to mirrors are rational multiples of π. Note that $\mathcal{Q}_n$ is a dense subset of $\mathcal{M}_n$, for $\mathcal{M}_n \cap \big(\mathbb{C} \times (0, \infty) \times e^{\pi i \mathbb{Q}}\big)^n$ is obviously a dense subset of $\mathcal{M}_n$, and is contained in $\mathcal{Q}_n$. Given an initial light-beam position, denote by $\mathcal{E}$ the set of all legal mirror configurations that allow at least one beam with that initial position to escape. Suppose we prove that, for every mirror configuration in $\mathcal{Q}_n$, at least one light beam with the given initial position escapes. Then $\mathcal{Q}_n \subseteq \mathcal{E}$, whence $\mathcal{E}$ is dense.

Assuming this density, we can also say something about the measure of $\mathcal{E}$. From the Lebesgue measure on $\big(\mathbb{R}^2 \times (0, \infty) \times [0, 2\pi)\big)^n$, construct a measure on $\mathcal{M}_n$ in the obvious way. Then, since E is open, by Corollary 3-8, E has positive measure. Furthermore, since E is both dense and open, its intersection with any nonempty open subset of $\mathcal{M}_n$ is open and nonempty; hence, this intersection also has positive measure. For any bounded open subset F of $\mathcal{M}_n$, we can normalize our measure on $\mathcal{M}_n$ to get a probability space on F. The measure of $E \cap F$ in nonzero in this space. We have just proved the following theorem.

Theorem 4-1. *Suppose that, for a given initial position, all rational legal mirror configurations let at least one light beam with that initial position escape. Then a mirror configuration randomly chosen from a bounded open subset of $\mathcal{M}_n$ has a nonzero probability of letting a light beam with that initial position escape.*

Note that there is a sense in which the boundedness requirement in Theorem 4-1 is superfluous. For any given mirror configuration, we may perform a scale change without changing whether any light beam escapes, allowing us to assume that the mirrors are all contained in the unit disk centered about the origin. This assumption implies that all mirror configurations lie within a bounded subset of $\mathcal{M}_1^n$.

Our task for the rest of this paper is to prove the hypothesis of Theorem 4-1. Towards that end, we henceforth assume that M is a rational mirror configuration. Limiting ourselves to the rational case is immensely profitable primarily because, as shown in the next theorem, rational mirror configurations only allow a light beam to attain finitely many directions. This finiteness places many restrictions on light-beam dynamics.

Theorem 4-2. *There is a positive integer N_p and a map $\Gamma\colon S^1 \to \mathcal{P}(S^1)$ such that, for every direction p_0, the following conditions hold:*

1. $p(q_0, p_0, [0,\infty)) \subseteq \Gamma(p_0)$ *for all* $q_0 \in \mathbb{C}$,
2. $\Gamma(v) = \Gamma(p_0)$ *for all* $v \in \Gamma(p_0)$, *and*
3. $|\Gamma(p_0)| \le N_p$.

Proof: Since M is rational, for each $j = 1, \ldots, n$, we have $\theta(M_j)\overline{\theta(M_1)} = e^{\pi i r_j}$ for some rational number r_j. For each r_j, choose an integer a_j and a positive integer b_j such that $r_j = a_j/b_j$. Let b be the least common multiple of $b_1, \ldots, b_n$. Because of how we defined b, for every j, there is an integer c_j such that $a_j/b_j = c_j/b$. Let $p_0 \in S^1$. Define the set $\Gamma(p_0)$ by

$$\Gamma(p_0) = \left\{e^{2\pi i j/b}\theta(M_1)^2\overline{p}_0 : 0 \le j \le b-1\right\} \cup \left\{e^{2\pi i j/b}p_0 : 0 \le j \le b-1\right\}. \tag{4-1}$$

Clearly $\Gamma(p_0)$ contains at most $2b$ elements. Moreover, if $v \in \Gamma(p_0)$, then $\Gamma(v) = \Gamma(p_0)$. All that remains is to show that a light beam with initial direction p_0 only attains directions in $\Gamma(p_0)$. Since $p_0 \in \Gamma(p_0)$, we only need to show that any light beam, with a direction currently in $\Gamma(p_0)$, does not attain a direction outside of $\Gamma(p_0)$ after being reflected off of a mirror.

A light beam that hits a mirror with direction v leaves that mirror with direction $\theta(M_j)^2\overline{v}$ provided j is the index of the mirror hit. Hence, we only need to show that $\theta(M_j)^2\overline{v} \in \Gamma(p_0)$ for $j = 1, \ldots, n$ and for all $v \in \Gamma(p_0)$. Therefore, choose an arbitrary k from $\{0, \ldots, b-1\}$. If $v = e^{2\pi i k/b}p_0$, then

$$\theta(M_j)^2\overline{v} = e^{2\pi i(c_j - k)/b}\theta(M_1)^2\overline{p}_0 \in \Gamma(p_0).$$

Likewise, if $v = e^{2\pi i k/b}\theta(M_1)^2\overline{p}_0$, then

$$\theta(M_j)^2\overline{v} = e^{2\pi i(c_j - k)/b}p_0 \in \Gamma(p_0). \qquad \square$$

For a more precise analysis of light-beam dynamics, we augment our concept of light-beam state with parity. Define the set of *parities* to be $\{\pm 1\}$. Take even integers to have parity 1 and odd integers to have parity -1. The *initial parity* of a light beam is arbitrary. The *parity* of a light beam, at a given time, is the product of its initial parity and the parity of the number of times the light beam has hit mirrors.

We denote light-beam parity by ξ with the symbolic definition given below in the case where the initial parity is denoted by ξ_0:

$$\xi\colon \mathbb{C} \times S^1 \times [0,\infty) \times \{\pm 1\} \to \{\pm 1\}$$

$$\xi(q_0, p_0, t, \xi_0) = \xi_0(-1)^{\max\{i : \tau_i(q_0,p_0) \le t\}}.$$

Set $\langle q, p\rangle(q_0, p_0, t, \xi_0) = \langle q, p\rangle(q_0, p_0, t)$. We call the ordered triple $\langle q, p, \xi\rangle(q_0, p_0, t, \xi_0)$ the *augmented state* of a light beam at time t provided that the light beam's initial augmented state is $\langle q_0, p_0, \xi_0\rangle$.

Given the definition of augmented state, we can augment our results on time-reversed determinism. Suppose that $\langle q_0, p_0, \xi_0 \rangle$ is an augmented light-beam state, t is a nonnegative real, and the light beam with initial state $\langle q_0, p_0 \rangle$ is not absorbed at or prior to time t. Set

$$\langle q_0', p_0', \xi_0' \rangle = \langle q, p, \xi \rangle (q_0, p_0, t, \xi_0).$$

If $s \in [0, t] - \{\tau_i(q_0, p_0) : i \geq 1\}$ and q_0' is not on a mirror or mirror endpoint, then

$$\langle q, p, \xi \rangle (q_0, p_0, s, \xi_0) = \langle q, -p, \xi \rangle (q_0', -p_0', t - s, \xi_0'). \tag{4-2}$$

To prove (4-2), simply count reflections.

Since there are only finitely many mirrors and parities, we have the following corollary to Theorem 4-2.

Corollary 4-3. *For every trapped light beam, there is a mirror M_i, a direction v, and a parity η such that the light beam hits the mirror M_i, with exiting direction v and exiting parity η, infinitely many times.*

Recalling the abbreviations (3-3) and (3-4), for all $i \geq 0$, let x_i be the unique real satisfying $q_i = \zeta_i + x_i \theta_i$. Then $x_i = \overline{\theta}_i (q_i - \zeta_i)$. Also, assuming that ξ_0 has been defined, let $\xi_i = (-1)^i \xi_0$. From Lemma 3-6 and some algebra we omit, it follows that

$$x_{i+1} = -x_i \frac{\operatorname{Im}(\overline{p}_i \theta_i)}{\operatorname{Im}(\overline{p}_{i+1} \theta_{i+1})} + \frac{\operatorname{Im}(\overline{p}_i (\zeta_{i+1} - \zeta_i))}{\operatorname{Im}(\overline{p}_{i+1} \theta_{i+1})}. \tag{4-3}$$

Proposition 4-4. *Let i and j be positive integers satisfying $i < j$. If $\xi_i = \xi_j$, if $\theta_i = \theta_j$, and if $p_i = p_j$, then x_j as a function of x_i is a translation.*

Proof: Since $\xi_i = \xi_j$, the difference $j - i$ is even. Therefore, we have

$$\frac{\partial x_j}{\partial x_i} = \prod_{i \leq k < j} \frac{\partial x_{k+1}}{\partial x_k} = (-1)^{j-i} \frac{\operatorname{Im}(\overline{p}_i \theta_i)}{\operatorname{Im}(\overline{p}_j \theta_j)} = 1. \qquad \square$$

Note that Proposition 4-4 is still be true for irrational mirror configurations. This generality allows us to prove Theorem 3-5:

Proof of Theorem 3-5: For the duration of this proof, let M be a possibly irrational legal mirror configuration. Suppose $\langle q_0, p_0 \rangle$ has a hit sequence with period P. Then $\zeta_{i+P} = \zeta_i$ and $\theta_{i+P} = \theta_i$ and $\lambda_{i+P} = \lambda_i$ for $i \geq 1$; so, we can apply Lemma 3-6 to construct a map f such that $\langle q_{1+(i+1)P}, p_{1+(i+1)P} \rangle = f(q_{1+iP}, p_{1+iP})$ for $i \geq 0$. Moreover, by Theorem 3-1, $p_{i+P} = p_i$ for $i \geq 1$; hence, by (4-3), f induces a map h such that $x_{1+(i+1)P} = h(x_{1+iP})$ for $i \geq 0$. Clearly $\xi_{1+2iP} = \xi_1$ for $i \geq 0$. Therefore, by Proposition 4-4, $h \circ h$ is a translation. If this translation is the identity, then the light beam with initial state $\langle q_0, p_0 \rangle$ has periodic state. If this translation is not the identity, then, for N sufficiently large, $x_{1+2NP} \notin (0, \lambda_1) = (0, \lambda_{1+2NP})$, which is absurd. $\square$

Proposition 4-4 also allows us to use the rationality of M to reduce the problem of understanding the dynamics of single light beam, which has a three-dimensional phase space, to a much simpler one-dimensional dynamical problem. Suppose the light beam with initial state $\langle q_0, p_0 \rangle$ is trapped. Then it hits some mirror, say the mirror M_k, infinitely many times. Furthermore, it hits that mirror infinitely many times at the same exiting direction and parity. Therefore, we focus our attention on the sequence of offsets on the mirror M_k that the trapped light beam hits infinitely many times at some fixed exiting direction and parity. These positions are specified by a subsequence of $\langle x_i \rangle_{i=0}^{\infty}$. Proposition 4-4 tells us that a given offset in this subsequence is determined from its predecessor by a translation that depends only on the intervening sequence of mirror hits. Usign this information, we construct a piecewise translation T, defined on part of $(0, \lambda(M_k))$. We prove more than a few lemmas concerning T, all of them means to the end of proving the following powerful theorem.

Theorem 4-5. *Let $\langle q_0, u_0 \rangle$ be the initial state of a trapped light beam. Then some element of $\Gamma(u_0)$ is degenerate or not aperiodic.*

Before we can construct T, we need some preliminary results. Consider the fate the light beam with initial augmented state $\langle \zeta(M_k) + x\theta(M_k), p_0, \xi_0 \rangle$ where $x \in (0, \lambda(M_k))$. In the next lemma, we show that, for all but finitely many values of x, the light beam does not hit a mirror endpoint without first hitting the mirror M_k with exiting direction and parity $\langle p_0, \xi_0 \rangle$.

First, we need some more terminology. For all x in $(0, \lambda(M_k))$, we say that x *returns* to $\langle M_k, p_0, \xi_0 \rangle$ if, at some positive time t, there exists a y in $(0, \lambda(M_k))$ such that we have

$$\langle q, p, \xi \rangle(\zeta(M_k) + x\theta(M_k), p_0, t, \xi_0) = \langle \zeta(M_k) + y\theta(M_k), p_0, \xi_0 \rangle.$$

We also abbreviate "returns to $\langle M_k, p_0, \xi_0 \rangle$" with just "returns" when doing so causes no ambiguity. Finally, we say that x escapes, hits a mirror, hits a mirror endpoint, is trapped, etc., if the light beam with initial state $\langle \zeta(M_k) + x\theta(M_k), p_0 \rangle$ escapes, hits a mirror, hits a mirror endpoint, is trapped, etc.

Lemma 4-6. *Only finitely many offsets in $(0, \lambda(M_k))$ hit a mirror endpoint without first returning.*

Proof: Let z be a mirror endpoint position, v be a direction, and η be a parity. If the light beam with initial augmented state $\langle z - v\delta(M)/2, -v, \eta \rangle$ hits mirror M_k with incoming direction and parity $\langle -p_0, \xi_0 \rangle$, then let w be the first point on mirror M_k so hit. Therefore, if $-v$ does not point from z into the mirror of the endpoint z, then (4-2) implies that w is the only offset on the mirror M_k where the light beam with initial augmented state $\langle w, p_0, \xi_0 \rangle$ hits z with incoming direction and parity $\langle v, \eta \rangle$ without first returning. Moreover, if w exists, then $v \in \Gamma(p_0)$.

On the other hand, if $-v$ points from z into the mirror of the endpoint z, then it is impossible to hit z with incoming direction v. Therefore, since there are finitely many mirror endpoints, finitely many directions in $\Gamma(p_0)$, and finitely many parities, there are only finitely many offsets in $(0, \lambda(M_k))$ that hit a mirror endpoint without first returning. $\square$

Set $m - 1$ equal to the number of offsets in $(0, \lambda(M_k))$ that hit a mirror endpoint without first returning. Label these offsets $\alpha_1, \ldots, \alpha_{m-1}$ such that $\alpha_{i-1} < \alpha_i$ for $i = 1, \ldots, m-1$. Also, set $\alpha_0 = 0$ and $\alpha_m = \lambda(M_k)$. Then, for $i = 1, \ldots, m$, define I_i as (α_{i-1}, α_i). Therefore, all the elements in a given interval I_i either return or never hit a mirror endpoint. We say that I_i is a *returning interval* if an element of I_i returns at least once.

Starting with the next lemma, let us use the term "hit sequence" in several new ways, using context to disambiguate its meaning. First, given an offset x on the mirror M_k, let *the hit sequence of x* denote the hit sequence of the light beam with initial state $\langle \zeta(M_k) + x\theta(M_k), p_0 \rangle$. Second, call the sequence of mirrors indices by which an offset returns the *hit sequence by which it returns*. Also, we say that two hit sequences *disagree* if, for some $k > 0$, their kth elements are distinct and nonzero, recalling that a zero in a hit sequence indicates escape or hitting a mirror endpoint. Conversely, we say that two sequences of elements of $\{0, \ldots, n\}$ *agree* if the nonzero part of one of the sequences contains the nonzero part of the other.

Lemma 4-7. *If $1 \leq i \leq m$ and I_i is a returning interval, then all elements of I_i return at least once, making their first return by the same hit sequence.*

Proof: Suppose I_i contained an offset x that returned at least once, with its first return occurring after N mirror hits. Then, by Corollary 3-10, x is contained in an open interval J of offsets that return at least once by the same hit sequence that x first returns. Choose the largest such J. Then J does not contain α_{i-1} or α_i; hence, J is contained in I_i.

Consider a boundary point y of J. The hit sequence of y, within the first N hits, either terminates or disagrees with the hit sequence of x. By Corollary 3-10, a disagreement or a termination due to escape implies that y is contained in an open set disjoint from J. Hence, the hit sequence of y must terminate due to hitting a mirror endpoint. Hence, y is a boundary point of I_i. Therefore, $J = I_i$. □

Define the operator T as the map from the union of the returning intervals into $(0, \lambda(M_k))$ with returning offsets mapped to the offsets at which they first return. Let us state some basic properties of T. From time-reversed determinism, it immediately follows that T is injective. Also, since offsets in a returning interval all return by the same finite hit sequence, Proposition 4-4 applies. Hence, the whole interval, when it first returns, is translated by the same amount. Therefore, T is a translation when restricted to any one of the returning intervals; hence, the range of T is also a finite union of open intervals. Therefore, if A is a subset of $(0, \lambda(M_k))$, then TA and $T^{-1}A$ are finite unions of open intervals if A is. It follows by induction that $T^{-i}\mathrm{dom}(T)$ and $T^i\mathrm{ran}(T)$ are finite unions of open intervals for all nonnegative integers i. Furthermore, T^{i+1} is a translation when restricted to any open interval contained in $T^{-i}\mathrm{dom}(T)$. Likewise, T^{-i-1} is a translation when restricted to any open interval contained in $T^i\mathrm{ran}(T)$.

Since T is a piecewise translation on a one-dimensional space, it is much easier to investigate iterating T than to directly investigate repeated reflections of light beams. We digress to point out that this approach is also profitable in the problem of polygonal billiards, which is equivalent to our problem, save that the table edges (mirrors) touch to form a closed polygon. Boldrighini *et al.*[**2**, p. 539] use the constraint of rationality to prove ergodicity results about T, and translate them into ergodicity results about the billiard dynamics. They actually prove a result similar to Theorem 4-5: a billiard, provided it is initially inside a rational polygon and has a nondegenerate initial direction, attains a set of positions that is dense in the interior of the polygon. Unfortunately, the similarities end here. We cannot apply ergodic thoery to our problem because T is only a partial function on $(0, \lambda(M_k))$. Fortunately, we can still prove plenty about T using elementary methods.

Proposition 4-8. *For all $i \geq 1$, the boundary points of* $\mathrm{dom}(T^i)$ *must eventually hit endpoints.*

Proof: If an offset x returns i times, then it is in $\mathrm{dom}(T^i)$, which does not contain any of its boundary points. If x escapes before returning i times, then, for some $j < i$, we have $x \in \overline{\mathrm{dom}(T^j)}$, but $T^j x \notin \overline{\mathrm{dom}(T)}$, as $T^j x$ is in one of the nonreturning intervals. Hence, $x \notin \overline{\mathrm{dom}(T^{j+1})}$; hence, $x \notin \overline{\mathrm{dom}(T^i)}$. □

Next we define F, a generalization of T. The domain of F is X, defined by

$$X = \bigcup_{j=1}^{n} (0, \lambda(M_j)) \times \left(S^1 - \{\pm\theta(M_j)\}\right) \times \{j\}.$$

This space is just a representation of all the light-beam states lying on mirrors, except for light-beam states with directions parallel to the mirror they lie on. The representation is explicitly defined by the map ϕ, defined by

$$\phi\colon X \to \mathbb{C} \times S^1 \text{ given by } \phi(x, u, j) = \langle \zeta(M_j) + x\theta(M_j), u\rangle.$$

For every $w \in X$, if the light beam with initial state $\phi(w)$ hits at least one mirror, then define Fw as $\phi^{-1}(z)$ where $z = \langle q, p\rangle(\phi(w), \tau_1(\phi(w)))$. This definition makes the position component of z the first point at which the light beam with initial state $\phi(w)$ hits a mirror, and makes the direction component the direction at which the light beam exits the mirror after that hit. By reversed-time determinism, F is injective. Also, F^2 preserves parity, whereas F reverses parity. Hence, for all x in $\mathrm{dom}(T)$, the relationship between F and T is

$$Tx = \pi_1(F^{2j}x) \text{ where } j = \min\{h \geq 1 : F^{2h}(x, p_0, k) \in (0, \lambda(M_k)) \times \{p_0\} \times \{k\}\}.$$

We use F to prove a very useful property of T.

Lemma 4-9. *Suppose that $1 \leq i \leq m$ and I_i is a nonreturning interval. Then all offsets in I_i escape by the same hit sequence.*

Proof: Let σ be a sequence of mirror indices, and let N be the length of σ. Then, for any choice of σ, the set of offsets in I_i whose hit sequences agree with σ is open by Corollary 3-10. By the same corollary, the set of offsets in I_i that escape after N hits is open. Since I_i is nonreturning, none of its offsets hit mirror endpoints; hence, these open sets partition I_i. By connectivity, exactly one of these open sets is nonempty. Therefore, either all the offsets escape by the same hit sequence, or they are all trapped by the same hit sequence.

Suppose the latter. Then, for all integers r and s such that $r > s \geq 0$, the sets $F^{2r}(I_i \times \{p_0\} \times \{k\})$ and $F^{2s}(I_i \times \{p_0\} \times \{k\})$ are disjoint, for if this is not the case, then, since F is injective, $I_i \times \{p_0\} \times \{k\}$ intersects $F^{2(r-s)}(I_i \times \{p_0\} \times \{k\})$, in contradiction with our assumption that no offsets in I_i return. Moreover, since F is injective, $F^{2r+1}(I_i \times \{v\} \times \{k\})$ and $F^{2s+1}(I_i \times \{v\} \times \{k\})$ are also disjoint.

By Corollary 4-3, we may choose a mirror M_l, a direction $v' \in \Gamma(p_0)$, and a parity η' such that the offsets in I_i all hit the mirror M_l infinitely many times with exiting direction and parity $\langle v', \eta'\rangle$. Therefore, there exist a $j_0 \in \{0, 1\}$ and an infinite increasing sequence of natural numbers $\langle j_h\rangle_{h=1}^{\infty}$ such that

$$F^{j_0+2j_h}(I_i \times \{p_0\} \times \{k\}) \in (0, \lambda(M_l)) \times \{v'\} \times \{l\}$$

for $h > 0$. By Proposition 4-4, $\pi_1 \circ F^{2j_h}$ translates $\pi_1(F^{j_0}(I_i \times \{p_0\} \times \{k\}))$ for each $h > 0$. The unions of all these translates are contained in $(0, \lambda(M_l))$; hence, some of these translates must intersect, contrary to what we have already shown. □

Let J be an open subinterval of $(0, \lambda(M_k))$. We say that an offset in J *returns* to J if it returns to $\langle M_k, p_0, \xi_0\rangle$ and one of the offsets that it returns to is in J. Let T_J

map every offset in J that returns to J to the first offset at which it returns. Leave T_J undefined for all other offsets. By time-reversed determinism, T_J is injective. Also, the proof of Lemma 4-6 is easily modified to show that only finitely many offsets in J hit endpoints before returning to J. Also, by injectivity of T, at most two offsets in J return to $\langle M_k, p_0, \xi_0 \rangle$ at a boundary point of J before returning to J. If we take the complement, in J, of the finitely many points that, before returning to J, hit mirror endpoints or return to a boundary point of J, then we get a finite union of subintervals of J. Denote these intervals by $J_1, \ldots, J_l$. We also refer to them as the *J-classes*. Therefore, all offsets in J-classes, unless they return to J, do not hit a mirror endpoint or return at a boundary point of J.

The next lemma is the analogue of Lemma 4-7 for T_J.

Lemma 4-10. *Let J be an open subinterval of $(0, \lambda(M_k))$, and J_i be a J-class. If J_i contains an offset that returns to J, then all offsets in J_i return to J, making their first return by the same hit sequence.*

Proof: Let J_i be a J-class. Suppose J_i contains an offset x that returns to J at least once, with its first return occurring after N mirror hits. Then, by Corollary 3-10, x is contained in an open interval K of offsets that return to $\langle M_k, p_0, \xi_0 \rangle$ after making the same N mirror hits, in the same order, as x. There exists some positive integer h such that $T^h x$ is the first offset in J to which x returns. Since T^h is a translation on K, we may shrink K so that $T^h K \subseteq J$ and $x \in K$. Furthermore, we may then enlarge K to be the largest open interval of offsets that contains x and contains only offsets that return to J by the same N-mirror hit sequence as x. Then $K \subseteq J_i$, for K does not contain the boundary points of J_i.

Consider a boundary point y of K. If $y \in J_i$, then y does not hit a mirror endpoint or a boundary point of J before returning to J. Thus, if $y \in J_i$, then one of the following must occur:

1. The hit sequence of y terminates within the first N hits.
2. The hit sequence of y has a disagreement with the hit sequence of x within the first N hits.
3. The hit sequence of y is identical with that of x for the first N hits, but $T^h y \notin J$.

If there is a disagreement or a termination due to escape, then, by Corollary 3-10, y is contained in an open set disjoint from K. Hence, the hit sequence of y can only terminate due to hitting a mirror endpoint. But such a termination implies y is a boundary point of J_i. Therefore, Possibilities 1 and 2 cannot occur if $y \in J_i$. Suppose Possibility 3 occurs. Then T^h is a translation on a neighborhood of y as well as a translation on K. Since K must intersect this neighborhood, T^h identically translates y and all offsets in K; hence, $T^h y$ is a boundary point of $T^h K$. But $T^h y \notin J$ and $T^h K \subseteq J$; hence, $T^h y$ is a boundary point of J, which is absurd because y is in a J-class. Therefore, none of Possibilities 1, 2, or 3 occur; hence, $y \notin J_i$; hence, $K = J_i$. □

The analogue of Lemma 4-9 for T_J is also true.

Lemma 4-11. *Let J be an open subinterval of $(0, \lambda(M_k))$. Suppose that J_i is a J-class that contains no offsets that return to J. Then all offsets in J_i escape by the same hit sequence.*

Proof: Suppose that no offset in J_i escapes. No offset in J_i hits a mirror endpoint either; hence, by Lemma 4-9, J_i is contained in a returning interval I_j. Therefore, $J_i \subseteq \mathrm{dom} T$; hence, if an offset in TJ_i hits a mirror endpoint or escapes, then so does its preimage under T. Therefore, no offset in TJ_i hits a mirror endpoint or escapes. Again, by Lemma 4-9, TJ_i is contained in a returning interval. In general, if h is a positive integer, $T^h J_i$ is contained in a returning interval, and no offset of $T^h J_i$ hits a mirror endpoint or escapes, then $T^{h+1} J_i$ is contained in a returning interval, and no offset of $T^{h+1} J_i$ hits a mirror endpoint or escapes. By induction, $T_h J_i$ is contained in a returning interval for all $h \geq 0$; hence, T^h is a translation on J_i for all $h \geq 0$.

Let r and s be integers such that $0 \leq r < s$. Then $T^r J_i$ and $T^s J_i$ must be disjoint, for otherwise $T^{s-r} J_i$ intersects J_i, which is contained in J; this situation is absurd, for no offsets in J_i return to J. However, the intervals in the sequence $\langle T^h J_i \rangle_{h=0}^{\infty}$ all have the same length, yet they are contained in $(0, \lambda(M_k))$; hence, they are not pairwise disjoint; hence, our hypothesis, that no point in J_i escapes, is false. Furthermore, by the connectedness argument used in the proof of Lemma 4-9, all points in J_i escape by the same hit sequence. □

Lemma 4-10 and Lemma 4-11 immediately imply the following proposition.

Proposition 4-12. *Let J be an open subinterval of $(0, \lambda(M_k))$. Each offset in J returns to $\overline{J}$, escapes, or hits a mirror endpoint.*

Proposition 4-13. *Suppose x is a trapped offset in $(0, \lambda(M_k))$. Then x returns to every neighborhood of x.*

Proof: If J is a neighborhood of x, then J contains $(x - 1/N, x + 1/N)$ for some $N > 0$. By Proposition 4-12, x returns to $[x - 1/(N+1), x + 1/(N+1)]$. □

Using T, we define the related operators T_+ and T_-:

$$T_+ x = \lim_{y \searrow x} Ty, \text{ and } T_- x = \lim_{y \nearrow x} Ty.$$

The domains of these operators are exactly those offsets x for which the respective limits are well defined. Since T is an injective piecewise translation, so are T_+ and T_-. We say that an offset x is trapped *under* T_+, respectively, T_-, T_+^{-1}, T_-^{-1}, if $T_+^N x$, respectively, $T_-^N x$, $T_+^{-N} x$, $T_-^{-N} x$, is defined for all positive integers N. Likewise, we say an offset x returns to a set E under T_+, respectively, T_-, T_+^{-1}, T_-^{-1}, if $x \in E$ and there is a positive integer N such that $T_+^N x$, respectively, $T_-^N x$, $T_+^{-N} x$, $T_-^{-N} x$, is in E.

The next lemma is an analogue of Proposition 4-13.

Lemma 4-14. *Let $x \in [0, \lambda(M_k))$ and $\epsilon > 0$. If x is trapped under T_+, then x returns to $[x, x + \epsilon)$ under T_+. If x is trapped under T_-, then x returns to $(x - \epsilon, x]$ under T_-.*

Proof: Suppose x is trapped under T_+. Set $J = (x, x + \epsilon)$. Let J_1 be the leftmost J-class of J. If J_1 does not return to J then, by Lemma 4-11, there is a positive integer h such that T^h not defined on any element of J_1. However, $\inf J_1 = x$; hence, for all positive integers h, there is an $\epsilon' > 0$ such $(x, x + \epsilon')$ is in the domain of T^h. Therefore, by Lemma 4-10, all offsets in J_1 return to J by the same hit sequence. Hence, $T^h J_1 \subset J$ for some $h > 0$; hence, $T_+^h x \in J \cup \{x\}$. The rest of the lemma follows by symmetry. □

Proposition 4-15. *If x is a trapped offset in $(0, \lambda(M_k))$, then x returns to both $(x - \epsilon, x]$ and $[x, x + \epsilon)$ for all $\epsilon > 0$.*

Proof: Suppose $T^{i+1} x$ is undefined for some $i \geq 0$. Choose i to be as small as possible. Then $T^i x$ is defined. Since $T^i x$ does not hit a mirror endpoint, $T^i x$ is either in a returning interval or in a nonreturning interval. By Lemma 4-9, $T^i x$ is not in a nonreturning interval, hence, $T^{i+1}x$ is defined. Therefore, $T^i x$ is defined for all nonnegative integers i; hence, x is trapped under T_+ and T_-. □

The main advantage of T_+ and T_- over T is their symmetry with respect to time, as stated below.

Lemma 4-16. *Suppose $x \in [0, \lambda(M_k))$. Then x is trapped under T_+ if and only if it is trapped under T_+^{-1}. Likewise, if $x \in (0, \lambda(M_k)]$, then x is trapped under T_- if and only if it is trapped under T_-^{-1}.*

Proof: By symmetry, we need only prove the lemma's claim about T_+. We show that if x is trapped under T_+, then it is trapped under T_+^{-1}. The converse follows, again by symmetry.

Suppose x is trapped under T_+, but not under T_+^{-1}. Then there exists an $N > 0$ such that $T_+^{-N}x$ is undefined; hence, there exists an $\epsilon > 0$ such that T_+^{-N} is undefined on $[x, x+\epsilon)$, as T^{-N} is a local translation on its domain. By Lemma 4-14, there is a positive integer h such that $T_+^h x \in [x, x+\epsilon)$. Since $T_+^h x$ is trapped under T_+, there is also a positive integer h' such that $T_+^{h+h'}x \in [T_+^h x, x+\epsilon) \subseteq [x, x+\epsilon)$. In general, given a positive integer i such that $T_+^i x \in [x, x+\epsilon)$, there is a positive integer $j > i$ such that $T_+^j x \in [x, x+\epsilon)$. Therefore, we may assume $h > N$. Hence, T_+^{-N} is defined on $T_+^h x$; hence, $T_+^h x \notin [x, x+\epsilon)$, which is absurd. □

We just need one more lemma before we can prove Theorem 4-5.

Lemma 4-17. *Suppose p_0 is aperiodic and nondegenerate. Then*

$$0 = \min\{x \in [0, \lambda(M_k)) : x \text{ trapped under } T_+\}$$

provided this set is not empty. Likewise,

$$\lambda(M_k) = \max\{x \in (0, \lambda(M_k)] : x \text{ trapped under } T_-\}$$

provdided this set is not empty.

Proof: We prove the first of these two statements; the proof of the second follows by symmetry.

Set

$$A = \{x \in [0, \lambda(M_k)) : x \text{ trapped under } T_+\}.$$

Set $y = \inf A$. Suppose there exists a positive integer N such that $T_+^N y$ is undefined. Then there exists an $\epsilon > 0$ such that T^N is undefined on $(y, y+\epsilon)$; hence, $y+\epsilon$ is a lower bound for A, a contradiction. Hence, y is trapped under T_+. Hence, it suffice to prove $y = 0$.

Suppose $y > 0$. If y is trapped, then y returns to $(y-\epsilon, y]$ for every $\epsilon > 0$. Since p_0 is aperiodic, y cannot return to y, for if $T_+^N y = y$ for some $N > 0$, then there exists an $\epsilon' > 0$ such that $T^N z = z$ for all $z \in (y, y+\epsilon')$, in contradiction with the aperiodicity of p_0. Therefore, y returns to $(y-\epsilon, y)$. Thus, if y is trapped, then there exists an offset z such that $z < y$ and z is trapped. However, if z is trapped, then it is trapped under T_+, in contradiction with our definition of y. Therefore, y is trapped under T_+, but not trapped.

Since y is not trapped, there exists some $N \geq 0$ such that $y \in \mathrm{dom} T^N$ is defined but $T^N y$ is not in a returning interval. Since $y \in \mathrm{dom} T_+^{N+1}$, we cannot have $T^N y$ in a nonreturning interval. Therefore, y hits a mirror endpoint.

Consider the behavior of y in reversed time. Since y is trapped under T_+, it is trapped under T_+^{-1}. Is y trapped in reversed time? Suppose not. Then y eventually hits a mirror endpoint in reversed time. Since y also hits a mirror endpoint in forward time, the light-beam state $\langle \zeta(M_k) + y\theta(M_k),\ p_0\rangle$ is a path from a mirror endpoint to a mirror endpoint, contrary to our hypothesis that p_0 is nondegenerate. Therefore, y is trapped in reversed time.

By symmetry, Proposition 4-15 is true for reversed time. Therefore, in reversed time, y returns to $(y-\epsilon, y]$ for all $\epsilon > 0$. Since y does not return to y in forward time, it does not return to y in reversed time; hence, y returns to $(y-\epsilon, y)$ in reversed time. The preimages of y under T are trapped under T_+ because y is. Therefore, there exists an offset z such that $z < y$ and z is trapped under T_+, in contradiction with our definition of y. Therefore, $y = 0$. □

Finally, with all our knowledge about T and its cousins, we prove Theorem 4-5.

Proof of Theorem 4-5: Let $\langle q_0, u_0\rangle$ be the initial state of a trapped light beam. Let i be the index of the first mirror hit by $\langle q_0, u_0\rangle$, and let u be the corresponding exiting direction. Suppose that all elements of $\Gamma(u_0)$ are aperiodic, and that none are degenerate. We derive a contradiction.

First, we define the light-beam closure operator. Given a direction w and a mirror index h, let $\mathcal{L}(w,h)$ denote the set of all ordered pairs $\langle w', h'\rangle$ for which w' is a direction, h' is a mirror index, and there is a trapped light beam, with initial direction w and initial position on $g(M_h)$, that hits the mirror $M_{h'}$ with exiting velocity w'. Given a set F contained in $S^1 \times \{1,\ldots,n\}$ and an integer $h \geq 0$, set $\mathcal{L}_0 F = F$ and $\mathcal{L}_{h+1}F = \mathcal{L}(\mathcal{L}_h F)$. Define the *light-beam closure* of F to be $\bigcup_{h=0}^{\infty} \mathcal{L}_h F$, and denote it by $\mathcal{L}_\omega F$.

Given $F \subseteq S^1 \times \{1,\ldots,n\}$, define the *endpoint hull* of F, which we denote by $\mathcal{E}F$, to be the convex hull of the set of endpoints of mirrors with indices in $\pi_2(F)$. Since there are only finitely many mirror endpoints, we may choose $\langle v, j\rangle \in \mathcal{L}_\omega\{\langle u, i\rangle\}$ such that $\zeta(M_j)$ or $\zeta(M_j) + \lambda(M_j)\theta(M_j)$ is on the boundary of $\mathcal{E}\mathcal{L}_\omega\{\langle u, i\rangle\}$. Moreover, a trapped light beam hits at least two noncolinear mirrors; hence, $\mathcal{E}\mathcal{L}_\omega\{\langle u, i\rangle\}$ contains at least three noncolinear points; hence, we may assume $j \neq i$. Since $\langle v, j\rangle \in \mathcal{L}_\omega\{\langle v, j\rangle\} \subseteq \mathcal{L}_\omega\{\langle u, i\rangle\}$, the boundary of $\mathcal{E}\mathcal{L}_\omega\{\langle v, j\rangle\}$ contains $\zeta(M_j)$ or $\zeta(M_j) + \lambda(M_j)\theta(M_j)$. We may assume $\zeta(M_j)$ is on the boundary of $\mathcal{E}\mathcal{L}_\omega\{\langle v, j\rangle\}$, for the argument for the other case can be handled symmetrically.

To derive a contradiction, and thereby prove the theorem, we show that $\zeta(M_j)$ is in the interior of $\mathcal{E}\mathcal{L}_\omega\{\langle v, j\rangle\}$. First, define a map T' that is the same as T except that $\langle v, j, 1\rangle$ replaces $\langle p_0, k, \xi_0\rangle$. Likewise define T'_+ and T'_-. All that we have proved of T, T_+, and T_- is clearly also true of T', T'_+, and T'_-, respectively. In particular, since $\langle v, j\rangle \in \mathcal{L}_\omega\{\langle u, i\rangle\} - \{\langle u, i\rangle\}$, there exists an offset $y \in (0, \lambda(M_j))$ such that y is trapped, in the sense that the light beam with initial state $\langle \zeta(M_j) + y\theta(M_j),\ v\rangle$ is trapped, and hence trapped under T'_+. It is easily checked that $\pi_1(\mathcal{L}_\omega\{\langle u, i\rangle\}) \subseteq \Gamma(u) = \Gamma(u_0)$; hence, $v \in \Gamma(u_0)$. Hence, v is aperiodic and nondegenerate. Therefore, Lemma 4-17 implies 0 is the minimum offset trapped by T'_+.

Let us agree to abbreviate $\delta(M)$ by δ throughout this proof, for here we don't concern ourselves with mirror configurations other than M. Consider the light beam with initial state $\langle \zeta(M_j) + v\delta/2,\ v\rangle$. If it escapes, then, by Theorem 3-9, it has an open neighborhood of initial states for which the corresponding light beams escape by the same hit sequence. For some $\epsilon > 0$, all $z \in [0, \epsilon)$ are such that $\langle \zeta(M_j) + z\theta(M_j) + v\delta/2,\ v\rangle$ is contained in this neighborhood. By definition of δ, the light beam with initial state $\langle \zeta(M_j) + z\theta(M_j) + v\delta/2,\ v\rangle$ has the same hit sequence as the light beam with initial state $\langle \zeta(M_j) + z\theta(M_j),\, v\rangle$. Therefore, $(T')^N$ is undefined on $(0, \epsilon)$ for some positive integer N. This situation is absurd, for 0 is trapped under T'_+.

Since it does not escape, the light beam with initial state $\langle \zeta(M_j) + v\delta/2,\, v\rangle$ is either trapped or hits a mirror endpoint. If the latter, then v is degenerate. Therefore, the light beam with initial state $\langle \zeta(M_j) + v\delta/2,\, v\rangle$ is trapped. Let k be the index of the first mirror hit by this light beam, and let x, p_0, and ξ_0 be the offset, exiting direction, and exiting parity, respectively, of this first hit. Note that $p_0 = \theta(M_k)^2\overline{v}$.

Let us show that $\langle p_0, k\rangle \in \mathcal{L}_\omega\{\langle v, j\rangle\}$. Since 0 is trapped under T'_+, it must return, under T'_+, to $[0, \epsilon)$ for all $\epsilon > 0$. Furthermore, since v is aperiodic, v must return under T'_+ to $(0, \epsilon)$ for all $\epsilon > 0$; hence, there exist arbitrarily small positive x such that x is trapped under T'_+. Furthermore, $(T')^h T'_+ 0$ is defined for all nonnegative integers h, for otherwise $T'_+ 0$ hits a mirror endpoint, in contradiction with the nondegeneracy of v. Therefore, we may choose $h > 0$ such that $(T'_+)^h 0$ is small enough that the light beams with initial states

$$\langle \zeta(M_j) + v\delta/2,\, v\rangle \text{ and } \langle \zeta(M_j) + (T'^h_+ 0)\theta(M_j) + v\delta/2,\, v\rangle$$

hit the same mirror first. Since $(T'_+)^h 0$ is defined for all $h > 0$, the light beam with initial state $\langle \zeta(M_j) + (T'^h_+ 0)\theta(M_j),\, v\rangle$ is trapped; hence, $\langle p_0, k\rangle \in \mathcal{L}_\omega\{\langle v, j\rangle\}$ as desired.

Since the light beam with initial state $\langle \zeta(M_j) + v\delta/2,\, v\rangle$ is trapped, x must be trapped under T where as usual T is defined with respect to $\langle M_k, p_0, \xi_0\rangle$. Moreover, p_0 is aperiodic because $p_0 \in \Gamma(u_0)$; hence, x returns to both $(x - \epsilon, x)$ and $(x, x + \epsilon)$ for all $\epsilon > 0$. Therefore, for any $\sigma \in \{\pm 1\}$, there exist arbitrarily small positive ϵ such that x returns at $x + \sigma\epsilon$. By definition of T, the light beam with initial augmented state $\langle \zeta(M_k) + x\theta(M_k), p_0, \xi_0\rangle$ hits $\langle \zeta(M_k) + (x + \sigma\epsilon)\theta(M_k), p_0, \xi_0\rangle$ in finite time.

Let us abbreviate $\zeta(M_k) + x\theta(M_k)$ by κ, and $\zeta(M_k) + (x + \sigma\epsilon)\theta(M_k)$ by α. Figure 4-1 illustrates κ, α, and several symbols we will define shortly. Since $p_0 = \theta(M_k)^2\overline{v}$, the light beam with initial state $\langle \alpha, p_0\rangle$ must exit the mirror M_k with direction $-v$ in reversed time. Let l be the index of the first mirror this light beam hits in reversed time. Then the mirror M_l must be hit, in forward time, by the light beam with initial state $\langle \kappa, p_0\rangle$ with exiting direction v. Therefore, $\langle v, l\rangle \in \mathcal{L}_\omega\{\langle k, p_0\rangle\} \subset \mathcal{L}_\omega\{\langle v, j\rangle\}$. Therefore, by convexity, $g(M_j)$, $g(M_k)$, and $g(M_l)$ are contained in $\mathcal{E}\mathcal{L}_\omega\{\langle v, j\rangle\}$. Furthermore, any points on lines segments with both segment endpoints in $g(M_j) \cup g(M_k) \cup g(M_l)$ must also be in $\mathcal{E}\mathcal{L}_\omega\{\langle v, j\rangle\}$.

Let α' be the first point on the mirror M_l hit by $\langle \alpha, p_0\rangle$ in reversed time. Set $\tau = |\alpha - \alpha'|$. Then, for all $\tau' \in [0, \tau]$, we have $\alpha - \tau' v \in \mathcal{E}\mathcal{L}_\omega\{\langle v, j\rangle\}$; hence, it suffices to show that, for some $\tau' \in [0, \tau]$, the point $\zeta(M_j)$ is in the interior of the convex hull of the three points $\zeta(M_j) + \lambda(M_j)\theta(M_j)$, α, and $\alpha - \tau' v$.

Set $t = |\zeta(M_j) - \kappa|$ and $\beta = \zeta(M_j) + \sigma\epsilon\theta(M_k)$. Then $\kappa = \zeta(M_j) + tv$ and $\alpha = \beta + tv$. Also, choose σ as follows:

$$\sigma = -\mathrm{sign}\left(\frac{\mathrm{Im}(\theta(M_k)\overline{v})}{\mathrm{Im}(\theta(M_j)\overline{v})}\right).$$

Lemma 4-18. *There exist a unique $r \in \mathbb{R}$ and a unique $s \in \mathbb{R}$ such that*

$$\zeta(M_j) - r\theta(M_j) = s\alpha + (1 - s)\beta.$$

Moreover, $r > 0$.

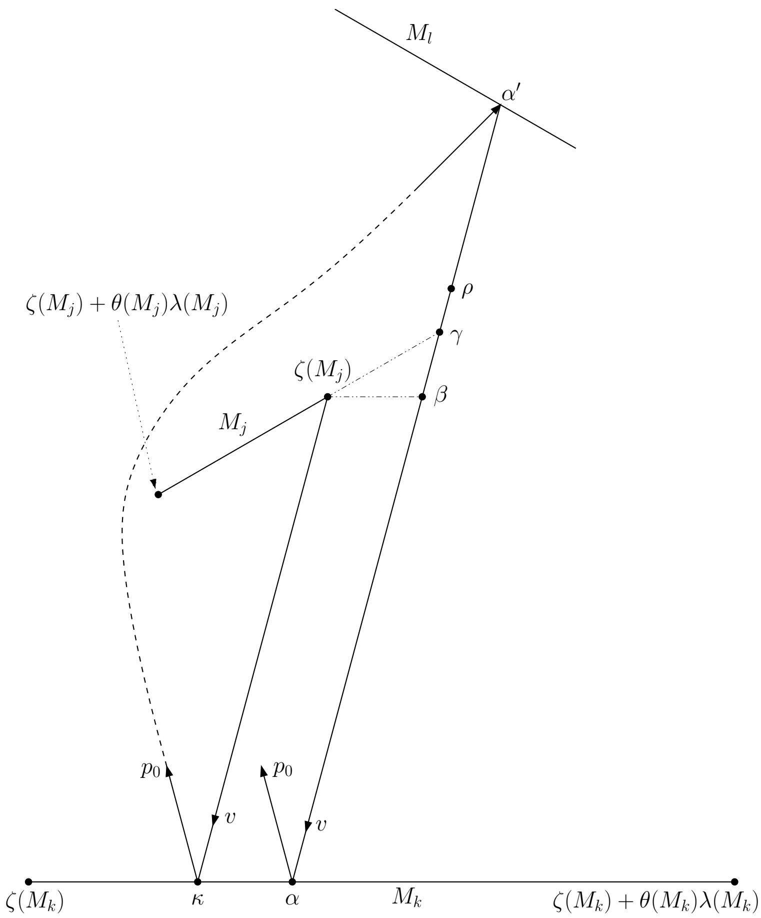

Figure 4-1. Important points and directions used in the proof of Theorem 4-5. The interior of the convex hull of the points α, ρ, and $\zeta(M_j) + \lambda(M_j)\theta(M_j)$ contains $\zeta(M_j)$.

Proof: The above equation is equivalent to $\zeta(M_j) - r\theta(M_j) - \beta = s(\alpha - \beta) = stv$. And it is equivalent to

$$st = \overline{v}(\zeta(M_j) - r\theta(M_j) - \beta) = -\overline{v}(r\theta(M_j) + \sigma\epsilon\theta(M_k)).$$

Since the rightmost expression must have zero imaginary part, there is exactly one solution for r, and it is positive:

$$r = -\sigma\epsilon\frac{\mathrm{Im}(\theta(M_k)\overline{v})}{\mathrm{Im}(\theta(M_j)\overline{v})} = \epsilon\left|\frac{\mathrm{Im}(\theta(M_k)\overline{v})}{\mathrm{Im}(\theta(M_j)\overline{v})}\right|.$$

Note that $\mathrm{Im}(\theta(M_j)\overline{v})$ and $\mathrm{Im}(\theta(M_k)\overline{v})$ are nonzero because v is not parallel to the mirror M_j or the mirror M_k.

Given this solution for r, we have

$$st = \overline{v}\epsilon\sigma\left(\theta(M_j)\frac{\mathrm{Im}(\theta(M_k)\overline{v})}{\mathrm{Im}(\theta(M_j)\overline{v})} - \theta(M_k)\right).$$

It is equivalent to

$$\begin{aligned} st(\theta(M_j)\overline{v} - \overline{\theta(M_j)}v) &= \epsilon\sigma\left(\theta(M_j)\overline{v}(\theta(M_k)\overline{v} - \overline{\theta(M_k)}v) - \theta(M_k)\overline{v}(\theta(M_j)\overline{v} - \overline{\theta(M_j)}v)\right) \\ &= \epsilon\sigma(\theta(M_j)\overline{\theta(M_k)} + \theta(M_k)\overline{\theta(M_j)}). \end{aligned}$$

Therefore, s has a unique real solution:

$$s = \frac{\sigma\epsilon}{t}\frac{\mathrm{Re}(\theta(M_j)\overline{\theta(M_k)})}{\mathrm{Im}(\theta(M_j)\overline{v})}. \qquad \square$$

Set $\gamma = \zeta(M_j) - r\theta(M_j) = s\alpha + (1-s)\beta$ and $\rho = \gamma - v\delta/2$ and $\tau' = t(1-s) + \delta/2$. Then

$$\begin{aligned} \rho &= s\alpha + (1-s)\beta - v\delta/2 = \beta + stv - v\delta/2 \\ &= \alpha + (t(s-1) - \delta/2)\,v = \alpha - \tau' v. \end{aligned} \tag{4-4}$$

By shrinking ϵ if necessary, we may assume $s < 1$; hence, $\tau' > 0$. Moreover, $\tau' < \tau$, but this inequality takes a little more work to show.

By the definitions of k and κ, the closed line segment between $\zeta(M_j)$ and κ does not intersect the closure of any mirrors other than $g(M_j)$ and $g(M_k)$. Therefore, we may choose ϵ to be small enough that the closed line segment between β and α, which is just a translation of the former line segment by $\sigma\epsilon\theta(M_k)$, does not intersect the closure of any mirrors, except possibly $g(M_j)$ or $g(M_k)$. By Lemma 4-18, the line determined by α and β intersects the line determined by the mirror M_j exactly once, at $\zeta(M_j) - r\theta(M_j)$, which is not in $\overline{g(M_j)}$ because $r > 0$. Therefore, the closed line segment between β and α intersects only the closure of the mirror M_k. (Though not required for our argument, but required to justify Figure 4-1, we note that $l \neq j$, for the ray from α that goes through β hits $g(M_l)$ but not $g(M_j)$.)

By shrinking ϵ if necessary, we may assume

$$|\beta - \zeta(M_j)| = \epsilon < \delta/2 \text{ and } |\gamma - \zeta(M_j)| = r < \delta/2.$$

Therefore, by definition of δ and convexity of disks, the closed line segment between β and γ does not intersect the closure of any mirror, except possibly $g(M_j)$. Moreover, this line segment lies on the line determined by α and β; hence, this line segment does not intersect $\overline{g(M_j)}$. Set $\rho = \gamma - v\delta/2$. Then the closed line segment between γ and ρ lies on the line determined by α and β, for $\alpha - \beta = tv$. Hence, this line segment does not intersect $\overline{g(M_j)}$. Furthermore, we have

$$|\rho - \zeta(M_j)| \leq |\rho - \gamma| + |\gamma - \zeta(M_j)| < \delta.$$

Hence, by definition of δ and convexity of disks, the closed line segment between γ and ρ does not intersect the closure of any mirror other than $g(M_j)$.

The closed line segment between α and ρ is contained in the union of the three closed line segments with respective endpoints α, β and β, γ and γ, ρ. Therefore, the closed line segment between α and ρ intersects the closure of at most one mirror, namely the mirror M_k. Since v is not parallel to the mirror M_k, this intersection occurs at exactly one point, namely α. Therefore, the closed line segment between α and ρ is the initial part of the trajectory of a light beam with initial position α and initial direction $\pm v$. Moreover, the initial direction is $-v$, as $\rho = \alpha - \tau' v$ and $\tau' > 0$.

Since the initial direction is $-v$, the light beam with initial state $\langle \alpha, p_0 \rangle$ hits ρ in reversed time before it hits the mirror M_l in reversed time; hence, $\tau' < \tau$. Therefore, it suffices to show that $\zeta(M_j)$ is in the interior of the convex hull of the points α, ρ, and $\zeta(M_j) + \lambda(M_j)\theta(M_j)$, which is equivalent to finding three numbers $A, B, C \in (0, 1)$ such that

$$A + B + C = 1 \text{ and } \zeta(M_j) = A\alpha + B\rho + C(\zeta(M_j) + \lambda(M_j)\theta(M_j)).$$

To find such A, B, and C, first note that (4-4) implies

$$\gamma = \rho + \delta v/2 = \alpha + t(s-1)v \text{ and } \rho = \alpha + (t(s-1) - \delta/2)v.$$

Combining these equations, we get

$$\gamma = \frac{\delta/2}{t(1-s) + \delta/2}\alpha + \frac{t(1-s)}{t(1-s) + \delta/2}\rho. \tag{4-5}$$

Also, since $\gamma = \zeta(M_j) - r\theta(M_j)$, we have

$$\zeta(M_j) = \frac{r}{\lambda(M_j) + r}(\zeta(M_j) + \lambda(M_j)\theta(M_j)) + \frac{\lambda(M_j)}{\lambda(M_j) + r}\gamma. \tag{4-6}$$

Combining (4-5) and (4-6), we get

$$\begin{aligned}\zeta(M_j) = {} & \frac{r}{\lambda(M_j) + r}(\zeta(M_j) + \lambda(M_j)\theta(M_j)) + \frac{\lambda(M_j)}{\lambda(M_j) + r}\frac{\delta/2}{t(1-s) + \delta/2}\alpha \\ & + \frac{\lambda(M_j)}{\lambda(M_j) + r}\frac{t(1-s)}{t(1-s) + \delta/2}\rho.\end{aligned}$$

Take A, B, and C to be the respective coefficients for α, ρ, and $\zeta(M_j) + \lambda(M_j)\theta(M_j)$ in the above equation. □

Theorem 4-19. *Let A be the set of directions p_0 for which there exists a complex number q_0 such that the light beam with initial state $\langle q_0, p_0 \rangle$ is trapped. Then A is countable.*

Proof: Let $\langle q_0, p_0 \rangle$ be a light-beam state for which the light beam with initial state equal $\langle q_0, p_0 \rangle$ is trapped. Then, by Theorem 4-5, some element of $\Gamma(p_0)$ is not aperiodic or is degenerate. Suppose $v \in \Gamma(p_0)$ and v is not aperiodic. Then, by Theorem 4-2, $\Gamma(v) = \Gamma(p_0)$; whence, $p_0 \in \Gamma(v)$. By Corollary 3-2, the union of all $\Gamma(w)$ such that w is not aperiodic is countable, as each $\Gamma(w)$ is finite. Therefore, there are only countably many possibilities for p_0.

Suppose all of elements $\Gamma(p_0)$ are aperiodic. Then we may choose a degenerate direction v in $\Gamma(p_0)$. There are $4n^2$ ordered pairs of mirror endpoints, and, by Theorem 3-4, only countably many directions allow a path from one given mirror endpoint to another, not necessarily distinct, given mirror endpoint. Therefore, there are only countably many degenerate directions. By Theorem 4-2, $\Gamma(v) = \Gamma(p_0)$; hence, the set of initial directions p_0 such that $\Gamma(p_0)$ contains a degenerate element is the union of the finite sets $\Gamma(w)$ for which w is degenerate. This union is countable. □

Corollary 4-20. *For any given rational mirror configuration on n mirrors H and any initial position q_0, there are only countably many directions p_0 such that the light beam with initial state $\langle q_0, p_0 \rangle$ does not escape.*

Proof: First, there are only countably many initial directions for which there is a trapped light beam, for any initial position. Second, there are only countably many directions that allow a path from q_0 to a given mirror endpoint, and there are finitely many mirror endpoints; hence, there are only countably many initial directions for which a light beam with initial position q_0 hits a mirror endpoint. Finally, if q_0 lies on a mirror or a mirror endpoint, then there are at most two directions that point from q_0 into the mirror. □

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