

# NOETHERIAN TYPE IN TOPOLOGICAL PRODUCTS

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ABSTRACT. The cardinal invariant *Noetherian type*  $\text{Nt}(X)$  of a topological space  $X$  was introduced by Peregudov in 1997 to deal with base properties that were studied by the Russian School as early as 1976. We study its behavior in products and box-products of topological spaces.

We prove in Section 2:

- (1) There are spaces  $X$  and  $Y$  such that  $\text{Nt}(X \times Y) < \min\{\text{Nt}(X), \text{Nt}(Y)\}$ .
- (2) In several classes of compact spaces, the Noetherian type is preserved by the operations of forming a square and of passing to a dense subspace.

The Noetherian type of the Cantor Cube of weight  $\aleph_\omega$  with the countable box topology,  $(2^{\aleph_\omega})_\delta$ , is shown in Section 3 to be closely related to the combinatorics of covering collections of countable subsets of  $\aleph_\omega$ . We discuss the influence of principles like  $\square_{\aleph_\omega}$  and Chang's conjecture for  $\aleph_\omega$  on this number and prove that it is not decidable in ZFC (relative to the consistency of ZFC with large cardinal axioms).

Within PCF theory we establish the existence of an  $(\aleph_4, \aleph_1)$ -sparse covering family of countable subsets of  $\aleph_\omega$  (Theorem 3.20). From this follows an absolute upper bound of  $\aleph_4$  on the Noetherian type of  $(2^{\aleph_\omega})_\delta$ . The proof uses a methods that was introduced by Shelah in [33].

## 1. INTRODUCTION

We study a class of topological cardinal invariants which are obtained from the classical cardinal invariants weight,  $\pi$ -weight and character by means of the following order-theoretic definition.

**Definition 1.1.** [26] A poset  $(P, \leq)$  is  $\kappa^{op}$ -like for a cardinal  $\kappa$  if for every element  $p \in P$  the set  $\{x : p \leq x \in P\}$  has cardinality  $< \kappa$ . The *op-character* of a poset  $(P, \leq)$  is the least infinite cardinal  $\kappa$  for which  $(P, \leq)$  is  $\kappa^{op}$ -like.

**Definition 1.2.** [28]

- (1) The *Noetherian type* of a base  $\mathcal{B}$  is op-character of the partial order  $(\mathcal{B}, \subseteq)$ .
- (2) The *Noetherian type* of a topological space  $X$ , denoted  $\text{Nt}(X)$ , is the least Noetherian type of some base  $\mathcal{B}$  for the topology on  $X$ .
- (3) The  $\pi$ -*Noetherian type* of  $X$ , denoted  $\pi\text{Nt}(X)$ , is the least op-character of some  $\pi$ -base for  $X$ .

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- (4) The *local Noetherian type at the point  $x$* , denoted  $\chi\text{Nt}(x, X)$ , is the least op-character of a local base at  $x$ .
- (5) The *local Noetherian type of  $X$*  ( $\chi\text{Nt}(X)$ ) is  $\chi\text{Nt}(X) = \sup\{\chi\text{Nt}(x, X) : x \in X\}$ .

Spaces with Noetherian type  $\omega$  (respectively,  $\omega_1$ ) were called *Noetherian* (respectively *weakly Noetherian*) by Peregudov and Šapirovskii [29]. Spaces with countable Noetherian type were also studied under the name of *spaces with an Open in Finite (OIF) base* by Balogh, Bennett, Burke, Gruenhage, Lutzer and Mashburn in [6], by Bennett and Lutzer in [7] and by Bailey in [4], especially in the context of generalized metric spaces, metrization theorems and generalized ordered spaces.

### 1.1. Background and statement of results.

**Theorem 1.3.** [28, 26] *Let  $X = \prod_{i \in I} X_i$ . Then:*

$$\begin{aligned} \text{Nt}(X) &\leq \sup_{i \in I} \text{Nt}(X_i), \\ \pi\text{Nt}(X) &\leq \sup_{i \in I} \pi\text{Nt}(X_i), \\ \chi\text{Nt}(X) &\leq \sup_{i \in I} \chi\text{Nt}(X_i). \end{aligned}$$

All information about the Noetherian type of a space is lost in sufficiently large powers of the space. This is a consequence of the following theorem of Malykhin [25].

**Theorem 1.4.** *Let  $X = \prod_{i \in I} X_i$  where each  $X_i$  has a minimal open cover of size two (which is the case, for example, if  $X$  is  $T_1$  and has more than one point). If  $\sup_{i \in I} w(X_i) \leq |I|$ , then  $\text{Nt}(X) = \omega$ .*

In particular,  $\text{Nt}(X^{w(X)}) = \omega$  for every  $T_1$  space  $X$ .

Another easy, but nonetheless surprising consequence of the above theorem is the following.

**Example 1.5.** There are compact spaces  $X$  and  $Y$  such that  $\text{Nt}(X \times Y) < \text{Nt}(X) \cdot \text{Nt}(Y)$ .

*Proof.* Let  $\kappa$  be a regular infinite cardinal. Let  $X = 2^\kappa$ , with the usual topology and  $Y = \kappa + 1$  with the order topology. By Theorem 1.4 we have  $\text{Nt}(X \times Y) = \omega$ . However, it is easy to see using the Pressing Down Lemma that  $\text{Nt}(Y) = \kappa^+$ .  $\square$

In view of the above example it is natural to ask:

*Question 1.6.* Is it true that  $\text{Nt}(X^2) = \text{Nt}(X)$  for every (compact) space  $X$ ?

Balogh, Bennett, Burke, Gruenhage, Lutzer and Mashburn similarly asked whether there exists a space  $X$  with  $\text{Nt}(X^2) = \omega < \text{Nt}(X)$  (see [6], Question 1).

In Section 2 we offer some partial positive answers for the compact case, as well as an example of  $\text{Nt}(X \times Y) < \min\{\text{Nt}(X), \text{Nt}(Y)\}$ .

In Section 3 we study Noetherian type in spaces where  $G_\delta$  sets are open, and more generally, where  $\kappa$ -intersections of open sets are open, for  $\kappa \geq \aleph_0$ . We give a Noetherian analogue of a classical bound of Juhász on the cellularity of the  $G_\delta$  modification of a compact space. While Juhász's was a ZFC theorem, we assume

(a weakening of) the GCH and another condition in our result. However, we show, modulo large cardinals, that this result is sharp.

The Noetherian type of the Cantor Cube of weight  $\aleph_\alpha$  with the countable box topology,  $(2^{\aleph_\omega})_\delta$ , or, more generally, with the  $\aleph_m$ -box topology, is closely related to combinatorial properties of covering collections of countable subsets of  $\aleph_\alpha$ . The Noetherian type of the  $\aleph_m$ -box topology on  $2^{\aleph_n}$  is easily determined. The case of  $(2^{\aleph_\omega})_\delta$  is more interesting and is tightly connected to pcf theory. In the rest of Section 3 we apply Shelah's PCF theory to the task of estimating the Noetherian type in this space.

The exact value of  $\text{Nt}((2^{\aleph_\omega})_\delta)$  is undecidable in ZFC, the standard axiom system for set theory. However, an absolute upper bound of  $\aleph_4$  is obtained on it in ZFC (Corollary 3.21 below). This bound follows from a new PCF-theoretic result which has independent interest: there exists an  $(\aleph_4, \aleph_4)$ -sparse covering family of countable subsets of  $\aleph_\omega$  (Theorem 3.20 below). The proof of 3.20 uses methods that Shelah introduced into PCF theory in [33], a few years after his discovery of the  $\aleph_{\omega_4}$  bound on  $\text{cov}(\aleph_\omega, \aleph_0)$  [31].

## 2. SUBSETS OF BASES AND THE NOETHERIAN TYPE OF COMPACT SQUARES AND DENSE SUBSPACES

The only approach we know towards proving that  $\text{Nt}(X^2) = \text{Nt}(X)$  for a space  $X$  is based on the following lemma.

**Lemma 2.1.** ([27]) *Let  $X$  be any space and  $n \in \omega$ . Then  $\text{Nt}_{\text{box}}(X^n) = \text{Nt}(X)$ .*

Here  $\text{Nt}_{\text{box}}(X^n)$  is the least infinite cardinal  $\kappa$  such that  $X^n$  has a  $\kappa^{\text{op}}$ -like base consisting of boxes.

If we were able to prove that every base of  $X^n$  consisting of boxes contains a base which is  $\text{Nt}(X^n)^{\text{op}}$ -like, then  $\text{Nt}(X) = \text{Nt}_{\text{box}}(X^n) \leq \text{Nt}(X^n)$ , so we would be done because  $\text{Nt}(X^n) \leq \text{Nt}(X)$  by Theorem 1.3. Unfortunately, this is not true. A counterexample is offered by the irrationals, which, as shown by Corollary 2.3, have finite powers with bases consisting of boxes that do not contain  $\omega^{\text{op}}$ -like bases, despite all powers of the irrationals having Noetherian type  $\omega$ .

The following theorem, credited to Konstantinov in [2] (see also page 26 of [3]), partially answers Question 2 from [27].

**Theorem 2.2.** *The Baire space  $\omega^\omega$  (homeomorphic to the space  $\mathbb{P}$  of irrationals) has a base  $\mathcal{B}$  that lacks an  $\omega^{\text{op}}$ -like subcover (and hence contains no  $\omega^{\text{op}}$ -like base).*

*Proof.* For each  $s \in \omega^{<\omega}$  and  $n \in \omega$ , let  $U_{s,n}$  be the clopen set of all  $f \in \omega^\omega$  for which  $s \widehat{\ } i \subseteq f$  for some  $i \leq n$ . Let  $\mathcal{B}$  consist of the sets of the form  $U_{s,n}$ . This makes  $\mathcal{B}$  a base of  $\omega^\omega$ . Now suppose that  $\mathcal{A} \subseteq \mathcal{B}$  and  $\mathcal{A}$  is  $\omega^{\text{op}}$ -like. For each  $s \in \omega^{<\omega}$ , there can be at most finitely many  $U_{s,n} \in \mathcal{A}$ . Set  $t_0 = \emptyset$  and, given  $k < \omega$  and  $t_k \in \omega^{<\omega}$ , choose  $i_k \in \omega$  such that  $i_k > n$  for all  $U_{t_k,n} \in \mathcal{A}$ . Set  $t_{k+1} = t_k \widehat{\ } i_k$ . Set  $f = \bigcup_{k < \omega} t_k$ . If  $f \in U_{s,n}$  for some  $U_{s,n} \in \mathcal{A}$ , then  $s = t_k$  for some  $k$ , which implies that  $i_k \leq n$ , in contradiction with how we constructed  $f$ . Therefore,  $\bigcup \mathcal{A} \neq \omega^\omega$ .  $\square$

**Corollary 2.3.** *If  $X = \omega^\omega$ , then, for all  $\alpha \in [1, \omega_1)$ ,  $X^\alpha$  has a base  $\mathcal{B}$  consisting of boxes such that  $\mathcal{B}$  lacks an  $\text{Nt}(X^\alpha)^{\text{op}}$ -like subcover.*

*Proof.* Let  $p: \alpha \times \omega \leftrightarrow \omega$  and let  $h: \omega^\omega \cong (\omega^\omega)^\alpha$  be given by  $h(f)(i)(j) = f(p(i, j))$ . Observe that the  $h$ -image of every  $U_{s,n}$  from the proof of Theorem 2.2 is a box.

Therefore,  $X^\alpha$  has a base of boxes not containing an  $\omega^{\text{op}}$ -like subcover. Finally, observe that  $\text{Nt}(X^\alpha) = \text{Nt}(\omega^\omega) = \omega$  by Theorem 1.4.  $\square$

Whether every base of a metric space contains an  $\omega^{\text{op}}$ -like base is closely related to total metacompactness and total paracompactness.

**Definition 2.4.** A space  $X$  is *totally metacompact* (*totally paracompact*) if every base  $\mathcal{B}$  of  $X$  has a point-finite (locally finite) subcover  $\mathcal{A}$ .

Compact implies totally paracompact implies totally metacompact; less obviously, totally metacompact does not imply totally paracompact: Balogh and Bennett [5] noticed that Example 1 of [17] is a counterexample. (That counterexample is a Moore space; we do not know if there is a metrizable counterexample.) On the other hand, Lelek [22] showed that total metacompactness, total paracompactness, and the Menger property are equivalent in the context of separable metric spaces. The next theorem connects these covering properties with  $\omega^{\text{op}}$ -like bases.

**Definition 2.5.**

- A family  $\mathcal{F}$  of subsets of a space  $X$  is *open in finite*, or *OIF*, if every nonempty open set of  $X$  has at most finitely many supersets in  $\mathcal{F}$ .
- A space is *totally OIF* if every base has an OIF subcover.
- Let  $\text{bNt}(X)$  denote the least  $\kappa \geq \omega$  such that every base of  $X$  includes a  $\kappa^{\text{op}}$ -like base of  $X$ .

**Theorem 2.6.** *If  $X$  is a metric space, then  $\text{bNt}(X) = \omega$  if and only if  $X$  is totally OIF.*

*Proof.* If  $\text{bNt}(X) = \omega$ , then every base contains an  $\omega^{\text{op}}$ -like base, which is also an OIF subcover. Conversely, if  $\mathcal{A}$  is a base of  $X$  and  $\mathcal{B}_n$  is an OIF subcover of the elements of  $\mathcal{A}$  with diameter  $\leq 2^{-n}$ , for all  $n < \omega$ , then  $\bigcup_{n < \omega} \mathcal{B}_n$  is an  $\omega^{\text{op}}$ -like base.  $\square$

**Corollary 2.7.** *If  $X$  is a totally metacompact metric space, then  $\text{bNt}(X) = \omega$ .*

*Question 2.8.* Is there a metric space that has some but not all of the three properties totally OIF, totally metacompact, and totally paracompact?

**Corollary 2.9** (Lemma 2.9, [26]).  *$\text{bNt}(X) = \text{Nt}(X) = \omega$  for all compact metrizable  $X$ .*

*Question 2.10.* Is there a compact space  $X$  having a base that does not contain an  $\text{Nt}(X)^{\text{op}}$ -like base? In other words, is  $\text{bNt}(X) < \text{Nt}(X)$  possible for a compact  $X$ ?

Many non-compact metric spaces  $X$  satisfy  $\text{bNt}(X) = \text{Nt}(X)$  too. Every  $\sigma$ -locally compact metric space  $X$  is totally paracompact [10], so it satisfies  $\text{bNt}(X) = \text{Nt}(X) = \omega$ . (To be  $\sigma$ -locally compact is to be a countable union of closed subspaces that are each locally compact. It is not hard to show that a paracompact, locally  $\sigma$ -locally compact space is already  $\sigma$ -locally compact.) Indeed, every scattered metric space (even every C-scattered metric space) is totally paracompact (and  $\sigma$ -locally compact) [36].

*Remark.*  $\text{Nt}(X) = \omega$  for all metrizable  $X$ . Moreover, it was noted by Bennett and Lutzer in [7] that, “it is easy to prove that any metric space, and indeed any metacompact Moore space, has an OIF base.” Indeed, the proof would be an easy modification of the proof of Theorem 2.6: if  $\langle \mathcal{D}_n \rangle_{n < \omega}$  is a development, then, after

choosing a point-finite refinement  $\mathcal{R}_n$  of each  $\mathcal{D}_n$ , we obtain an OIF (and therefore  $\omega^{\text{op}}$ -like) base:  $\bigcup_{n < \omega} \mathcal{R}_n$ .

Returning our focus from metric spaces back to compacta, we prove next that  $\text{Nt}(X^2) = \text{Nt}(X)$  in several classes of compact spaces. Theorem 2.12 below handles the class of spaces  $X$  satisfying  $\chi(p, X) = w(X)$  for all  $p \in X$ . Further results show that, in particular, it is consistent that this holds for all homogeneous compacta.

**Proposition 2.11.** *If  $X$  is a space and  $\mathcal{A}$  is a  $(w(X)^+)^{\text{op}}$ -like base of  $X$ , then  $|\mathcal{A}| \leq w(X)$ .*

*Proof.* Seeking a contradiction, suppose that  $|\mathcal{A}| > w(X)$ . Let  $\mathcal{B}$  be a base of  $X$  of size  $w(X)$ . Every element of  $\mathcal{A}$  then contains an element of  $\mathcal{B}$ . Hence, some  $U \in \mathcal{B}$  is contained in  $w(X)^+$ -many elements of  $\mathcal{A}$ . Clearly  $U$  contains some  $V \in \mathcal{A}$ , so  $\mathcal{A}$  is not  $(w(X)^+)^{\text{op}}$ -like.  $\square$

We say that a space is  $\kappa$ -compact if every open cover has a subcover of size less than  $\kappa$ .

**Theorem 2.12** (Lemma 3.20, [26]). *Suppose that  $X$  is a space with no isolated points and  $\chi(p, X) = w(X)$  for all  $p \in X$ . Further suppose that  $\kappa = \text{cf}(\kappa) \leq \min\{\text{Nt}(X), w(X)\}$  and  $X$  has a network consisting of at most  $w(X)$ -many  $\kappa$ -compact sets. Every base of  $X$  then contains a  $\text{Nt}(X)^{\text{op}}$ -like base of  $X$ .*

*Remark.* If  $X$  is  $T_3$  and locally compact, then it is easily seen that  $X$  has a network consisting of at most  $w(X)$ -many compact sets.

The following two lemmas are easy modifications of Dow's Propositions 2.3 and 2.4 from [11].

**Lemma 2.13.** *Let  $X$  be a space with base  $\mathcal{A}$ ; let  $\kappa$  be an uncountable cardinal,  $\{X, \mathcal{A}, \kappa\} \subseteq M \prec H(\theta)$ , and  $\kappa \cap M \in \kappa + 1$ . Set  $B = \{p \in X : \text{ord}(p, \mathcal{A}) < \kappa\}$ . We then have  $\{U \in \mathcal{A} : p \in U\} \subseteq M$  for every  $p \in \overline{B \cap M}$ .*

*Proof.* Suppose that  $p \in \overline{B \cap M}$  and  $p \in U \in \mathcal{A}$ . Choose  $q \in U \cap B \cap M$ . Since  $\kappa \cap M \in \kappa + 1$ , we have

$$U \in \{V \in \mathcal{A} : q \in V\} \in [H(\theta)]^{<\kappa} \cap M \subseteq [M]^{<\kappa};$$

hence,  $U \in M$ .  $\square$

*Remark.* The conclusion of the above lemma immediately implies that  $\overline{B \cap M} \subseteq B$  if  $|M| < \kappa$  (but we do not use this fact).

**Lemma 2.14.** *Let  $X$  be a compact  $T_1$  space with base  $\mathcal{A}$  and let  $M$  be such that  $X, \mathcal{A} \in M \prec H(\theta)$  and  $\mathcal{A} \cap M$  includes a local base at every  $p \in \overline{X \cap M}$ . We then have  $\overline{X \cap M} = X$ ; hence,  $\mathcal{A} \cap M$  is a base of  $X$ .*

*Proof.* Seeking a contradiction, suppose that  $q \in X \setminus \overline{X \cap M}$ . Choose  $\mathcal{B} \subseteq \mathcal{A} \cap M$  such that  $q \notin \bigcup \mathcal{B} \supseteq \overline{X \cap M}$ . Choose a finite  $\mathcal{F} \subseteq \mathcal{B}$  such that  $\bigcup \mathcal{F} \supseteq \overline{X \cap M}$ . Since  $\mathcal{F} \in M$ , we have  $X \subseteq \bigcup \mathcal{F}$  by elementarity, in contradiction with  $q \notin \bigcup \mathcal{B}$ .  $\square$

**Theorem 2.15.** *Let  $X$  be a compact  $T_1$  space with base  $\mathcal{A}$  and let  $\kappa$  be a regular uncountable cardinal. Set  $B = \{p \in X : \text{ord}(p, \mathcal{A}) < \kappa\}$ . We then have  $w(\overline{B}) < \kappa$ .*

*Proof.* Choose  $M$  to be as in Lemma 2.13 and to have size less than  $\kappa$ . Applying Lemma 2.14 to the space  $\overline{B}$  and its base  $\mathcal{U} = \{U \cap \overline{B} : U \in \mathcal{A}\}$ , we get a sufficiently small base  $\mathcal{U} \cap M$  of  $\overline{B}$ .  $\square$

The following lemma improves upon Theorem 1 of Peregudov [28], which says that if  $X$  is a compactum, then  $w(X) \leq \pi\chi(X)\text{Nt}(X)$ , where  $\text{Nt}(X)$  is the supremum of all cardinals strictly below  $\text{Nt}(X)$ .

**Lemma 2.16.** *Let  $X$  be a compact space such that  $w(X) \geq \kappa$  where  $\kappa$  is some regular uncountable cardinal. If  $X$  has a dense set of points of  $\pi$ -character  $< \kappa$ , then  $\text{Nt}(X) > \kappa$ .*

*Proof.* Let  $\mathcal{B}$  be any base for  $X$ . By Theorem 2.15, there is an open set  $U \subset X$  such that every point of  $U$  has order at least  $\kappa$ . Let  $p \in U$  be a point of  $\pi$ -character less than  $\kappa$ , and  $\mathcal{C} \subset \mathcal{B}$  be a set such that  $|\mathcal{C}| = \kappa$  and  $p \in \bigcap \mathcal{C}$ . Since  $p$  has  $\pi$ -character less than  $\kappa$ , there is a nonempty open set that is in  $\kappa$ -many members of  $\mathcal{C}$ . So,  $\text{Nt}(X) > \kappa$ .  $\square$

The above lemma fails if  $\kappa$  is allowed to be singular.

**Example 2.17.** For one example, if  $Y$  is the one-point compactification of  $\bigoplus_{n < \omega} 2^{\omega_n}$ , then  $\pi\chi(p, Y) < \aleph_\omega = w(Y)$  for all  $p \in Y$ , yet  $\text{Nt}(Y) = \omega$  is witnessed by joining the canonical bases of  $2^{\omega_n}$  for  $n < \omega$  with  $\{Y \setminus \bigcup_{m < n} 2^{\omega_m} : n < \omega\}$ .

**Example 2.18.** For another example, let  $X = \prod_{n < \omega} A_{\aleph_n}$  where for all infinite cardinals  $\kappa$ ,  $A_\kappa$  denotes the one-point compactification  $D_\kappa \cup \{\infty\}$  of the discrete space  $D_\kappa$  with underlying set  $\kappa$ . Notice that  $w(X) = w(A_{\aleph_\omega}) = \aleph_\omega$  and  $\pi\chi(X) = \pi\chi(A_{\aleph_\omega}) = \omega$ . Let us show that  $\text{Nt}(A_{\aleph_\omega}) = \aleph_{\omega+1}$ , but  $\text{Nt}(X) = \aleph_\omega$ .

First, let us show that actually  $\text{Nt}(A_\kappa) = \kappa^+$  for all uncountable  $\kappa$ . Let  $\mathcal{U}$  be a base of  $A_\kappa$ . Set  $F = \{\sigma \subseteq \kappa : A_\kappa \setminus \sigma \in \mathcal{U}\} \in [[\kappa]^{<\omega}]^\kappa$ . Set  $T = \{\lambda^+ : \omega \leq \lambda < \kappa\}$  and let  $S$  be a cofinal subset of  $T$  of minimal order type. For each  $\mu \in S$ , choose  $I_\mu \in [F]^\mu$  such that  $I_\mu$  is a  $\Delta$ -system with root  $r_\mu$ . Partition each  $I_\mu$  into disjoint subsets  $J_\mu$  and  $K_\mu$  each of size  $\mu$ . Observe that if  $|S| < \kappa$ , then, setting

$$J = \bigcup_{\mu \in S} \left\{ \sigma \in J_\mu : \emptyset = (\sigma \setminus r_\mu) \cap \bigcup_{\nu \in \mu \cap S} K_\nu \right\},$$

$\bigcup J$  has size  $\kappa$  but does not equal  $\kappa$ . Moreover, if  $|S| = \kappa$ , then  $\kappa$  is weakly inaccessible, in which case  $F$  contains a  $\Delta$ -system  $G$  of size  $\kappa$ . For such a  $G$ , if  $\sigma \in G$ , then  $\bigcup(G \setminus \{\sigma\})$  has size  $\kappa$  but does not equal  $\kappa$ . Thus, in both cases there exists  $I \in [F]^\kappa$  such that  $\bigcap_{\sigma \in I} (A_\kappa \setminus \sigma)$  includes an isolated point. Hence,  $\mathcal{U}$  is not  $\kappa^{\text{op}}$ -like; hence,  $\kappa^+ \leq \text{Nt}(A_\kappa) \leq w(A_\kappa)^+ = \kappa^+$ .

Second, by Theorem 1.3,  $\text{Nt}(X) \leq \sup_{n < \omega} \text{Nt}(A_{\aleph_n}) = \aleph_\omega$ . Finally,  $\text{Nt}(X) \geq \aleph_\omega$  by Lemma 2.16.

**Example 2.19.** Building on the previous example, let  $Z$  be the one-point compactification of  $\bigoplus_{\alpha < \omega_1} A_{\aleph_\alpha}$ . Observe that  $w(Z) = w(A_{\aleph_{\omega_1}}) = \aleph_{\omega_1}$  and  $\pi\chi(Z) = \pi\chi(A_{\aleph_{\omega_1}}) = \omega$ . As argued above,  $\text{Nt}(A_{\aleph_{\omega_1}}) = \aleph_{\omega_1+1}$ . However, we will show that  $\text{Nt}(Z) = \aleph_{\omega_1}$ . First, by Lemma 2.16,  $\text{Nt}(Z) \geq \aleph_{\omega_1}$ . Second, we can build an  $\aleph_{\omega_1}^{\text{op}}$ -like base  $\mathcal{C}$  of  $Z$  as follows. For each  $\alpha < \omega_1$ , let  $\mathcal{A}_\alpha$  be (a copy of) a base of  $A_{\aleph_\alpha}$  of size  $\aleph_\alpha$ . Set  $\mathcal{B} = \{Z \setminus \bigcup_{\alpha \in \sigma} A_{\aleph_\alpha} : \sigma \in [\omega_1]^{<\omega}\}$ . Set  $\mathcal{C} = \mathcal{B} \cup \bigcup_{\alpha < \omega_1} \mathcal{A}_\alpha$ .

**Theorem 2.20.** *If  $X$  is a compactum with regular weight and  $X$  is homogeneous (or merely homogeneous in character), then every base of  $X$  contains an  $\text{Nt}(X)^{\text{op}}$ -like base.*

*Proof.* If  $\chi(X) = w(X)$ , then just apply Theorem 2.12. If  $\chi(X) < w(X)$ , then  $\text{Nt}(X) = w(X)^+$  by Lemma 2.16. So, if  $\mathcal{A}$  is any base for  $X$ , then every base of size  $w(X)$  contained in  $\mathcal{A}$  would be  $\text{Nt}(X)^{\text{op}}$ -like.  $\square$

We can exchange the above requirement that  $w(X)$  be regular for a weak form of GCH.

**Corollary 2.21.** *Suppose that every limit cardinal is strong limit. For every homogeneous compactum  $X$ , every base of  $X$  then contains an  $\text{Nt}(X)^{\text{op}}$ -like base.*

*Proof.* By Arhangel'skiĭ's Theorem,  $\chi(X) \leq w(X) \leq 2^{\chi(X)}$ . If  $\chi(X) < w(X)$ , then  $w(X)$  is a successor cardinal; apply Theorem 2.20. If  $\chi(X) = w(X)$ , then apply Theorem 2.12.  $\square$

**Corollary 2.22.** (GCH) *Let  $X$  be a homogeneous compactum. Then  $\text{Nt}(X^n) = \text{Nt}(X)$  for every  $n \in \omega$ .*

Geschke and Shelah [14] have shown that for every infinite cardinal  $\kappa \leq \mathfrak{c}$ , there is a first countable homogeneous compactum with weight  $\kappa$ . Since ZFC cannot refute  $\mathfrak{c} > \aleph_\omega$ , it is consistent to have a homogeneous compactum  $X$  such that  $\chi(X) = \omega = \text{cf}(w(X)) < w(X)$ . For such  $X$ , our above theorems do not determine whether  $\text{Nt}(X^2) = \text{Nt}(X)$ .

If we do not assume homogeneity, then we still have the following weak results.

**Theorem 2.23** (Lemma 3.23, [26]). *Suppose that  $\kappa = \text{cf}(\kappa) > \omega$  and  $X$  is a space such that  $\pi\chi(p, X) = w(X) \geq \kappa$  for all  $p \in X$ . Further suppose that  $X$  has a network consisting of at most  $w(X)$ -many  $\kappa$ -compact sets. Every base of  $X$  then contains a  $w(X)^{\text{op}}$ -like base of  $X$ .*

*Remark.* If  $X$  is  $T_3$  and locally compact, then it is easily seen that  $X$  has a network consisting of at most  $w(X)$ -many compact sets.

**Theorem 2.24.** *Suppose that  $\kappa$  is a regular cardinal and  $X$  is a locally  $\kappa$ -compact  $T_3$  space such that  $\text{Nt}(X) \leq w(X) = \kappa$ . Every base of  $X$  then contains a  $\kappa^{\text{op}}$ -like base of  $X$ .*

*Proof.* Let  $\mathcal{A}$  be a base of  $X$  and let  $\mathcal{B}$  be a  $\kappa^{\text{op}}$ -like base of  $X$ . We may assume that  $|\mathcal{A}| = |\mathcal{B}| = \kappa$ . Suppose that  $\kappa = \omega$ . The space  $X$  is then metrizable and  $\sigma$ -compact, so, as noted earlier for the wider class of  $\sigma$ -locally compact metric spaces, every base of  $X$  contains an  $\omega^{\text{op}}$ -like base.

Suppose that  $\kappa > \omega$ . Let  $\langle M_\alpha \rangle_{\alpha \leq \kappa}$  be a continuous elementary chain such that  $\{M_\beta : \beta < \alpha\} \cup \{\mathcal{A}, \mathcal{B}\} \subseteq M_\alpha \prec H(\theta)$  and  $|M_\alpha| < \kappa$  and  $M_\alpha \cap \kappa \in \kappa$  for all  $\alpha < \kappa$ . The inclusion  $\mathcal{A} \cup \mathcal{B} \subseteq M_\kappa$  follows immediately. For each  $\alpha < \kappa$ , let  $\mathcal{U}_\alpha$  denote the set of all  $U \in \mathcal{A} \cap M_{\alpha+1}$  for which  $U$  has a superset in  $\mathcal{B} \setminus M_\alpha$ . Set  $\mathcal{U} = \bigcup_{\alpha < \kappa} \mathcal{U}_\alpha \subseteq \mathcal{A}$ . First, let us show that  $\mathcal{U}$  is  $\kappa^{\text{op}}$ -like. Suppose that  $\alpha < \kappa$  and  $\mathcal{U}_\alpha \ni U \subseteq V \in \mathcal{U}$ . There then exist  $\beta < \kappa$  and  $B \in \mathcal{B} \setminus M_\beta$  such that  $B \supseteq V \in M_{\beta+1}$ . Hence,  $U \subseteq B$ ; hence,  $B \in \{W \in \mathcal{B} : U \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{<\kappa}$ ; hence,  $B \in M_{\alpha+1}$ ; hence,  $\beta \leq \alpha$ ; hence,  $V \in M_{\alpha+1}$ . Thus,  $\mathcal{U}$  is  $\kappa^{\text{op}}$ -like.

Finally, let us show that  $\mathcal{U}$  is a base of  $X$ . Suppose that  $p \in B \in \mathcal{B}$  and  $\overline{B}$  is  $\kappa$ -compact. It then suffices to find  $U \in \mathcal{U}$  such that  $p \in U \subseteq B$ . Let  $\beta$  be the least  $\alpha < \kappa$  such that there exists  $A \in \mathcal{A} \cap M_{\alpha+1}$  satisfying  $p \in A \subseteq \overline{A} \subseteq B$ .

Fix such an  $A \in \mathcal{A} \cap M_{\beta+1}$ . If  $B \notin M_\beta$ , then  $A \in \mathcal{U}_\beta$  and  $p \in A \subseteq B$ . Hence, we may assume that  $B \in M_\beta$ . For each  $q \in \overline{A}$ , choose  $\langle A_q, B_q \rangle \in \mathcal{A} \times \mathcal{B}$  such that

$q \in A_q \subseteq B_q \subseteq \overline{B}_q \subseteq B$ . There then exists  $\sigma \in [\overline{A}]^{<\kappa}$  such that  $\overline{A} \subseteq \bigcup_{q \in \sigma} A_q$ . By elementarity, we may assume that  $\langle \langle A_q, B_q \rangle \rangle_{q \in \sigma} \in M_{\beta+1}$ ; hence,  $A_q, B_q \in M_{\beta+1}$  for all  $q \in \sigma$ . Choose  $q \in \sigma$  such that  $p \in A_q$ . If  $B_q \notin M_\beta$ , then  $A_q \in \mathcal{U}_\beta$  and  $p \in A_q \subseteq B$ . Hence, we may assume that  $B_q \in M_\beta$ ; hence, we may choose  $\alpha < \beta$  such that  $B_q \in M_{\alpha+1}$ . It follows that  $B \in \{W \in \mathcal{B} : B_q \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{<\kappa}$ ; hence,  $B \in M_{\alpha+1}$ . For each  $r \in \overline{B}_q$ , choose  $W_r \in \mathcal{A}$  such that  $r \in W_r \subseteq \overline{W}_r \subseteq B$ . There then exists  $\tau \in [\overline{B}_q]^{<\kappa}$  such that  $\overline{B}_q \subseteq \bigcup_{r \in \tau} W_r$ . By elementarity, we may assume that  $\langle W_r \rangle_{r \in \tau} \in M_{\alpha+1}$ . Choose  $r \in \tau$  such that  $p \in W_r$ . We then have  $W_r \in \mathcal{A} \cap M_{\alpha+1}$  and  $p \in W_r \subseteq \overline{W}_r \subseteq B$ , in contradiction with the minimality of  $\beta$ . Thus,  $\mathcal{U}$  is a base of  $X$ .  $\square$

**Theorem 2.25.** *Let  $X$  be a compact space such that  $w(X)$  is a regular cardinal and  $X$  does not map onto  $I^{w(X)}$ . Then  $\text{Nt}(X^n) = \text{Nt}(X) = w(X)^+$  for every  $n \in \omega$ .*

*Proof.* By a well-known consequence of Šapirovskiĭ's Theorem on maps onto Tychonoff cubes (see [18], 3.20)  $X$  has a dense set of points of  $\pi$ -character  $< w(X)$ . But then also  $X^n$  has a dense set of points of  $\pi$ -character  $< w(X)$ . Therefore, by Lemma 2.16, we have  $w(X) = w(X^n) < \text{Nt}(X^n)$ . Let  $\mathcal{B}$  be a base for  $X^n$  of size  $w(X)$  consisting of boxes. Then  $\mathcal{B}$  is trivially  $\text{Nt}(X^n)^{op}$ -like base and hence we are done.  $\square$

**Corollary 2.26.**  *$\text{Nt}(X^n) = \text{Nt}(X)$  for every compact space such that  $w(X)$  is a regular cardinal and at least one of the following conditions holds:*

- (1)  $X$  is hereditarily normal.
- (2)  $\beta\omega$  does not embed in  $X$ .
- (3)  $|X| < 2^{w(X)}$ .

*Proof.* The case of the third item follows readily from Theorem 2.25. In case  $X$  is like in the first or the second item then  $X$  cannot even map onto  $I^{\omega_1}$  by the argument in the proof of 3.21 and 3.22 of [18].  $\square$

We now proceed to show the strongest instance of the failure of productivity of Noetherian type that we know of so far. Recall that a partial order is called *directed* if any two elements have a common upper bound. A map between partial orders is called *Tukey* if the images of unbounded sets are unbounded.

Let  $\kappa$  be a regular cardinal such that  $\kappa^{\aleph_0} = \kappa$  and let  $E_\omega^\kappa = \{\alpha < \kappa : cf(\alpha) = \omega\}$ . Order  $[\kappa]^{<\omega}$  with respect to inclusion. Let  $S_0$  and  $S_1$  be two stationary subsets of  $E_\omega^\kappa$  with nonstationary intersection. Let  $D_i = D(S_i)$  be the set of all countable compact subsets of  $S_i$ , ordered with respect to inclusion. The key facts we need about  $D_0$  and  $D_1$  are due to Todorćević:

**Theorem 2.27** (Lemma 2, [37]). *If  $\lambda$  is a regular uncountable cardinal and  $S$  and  $T$  are unbounded subsets of  $E_\omega^\lambda$ , then there is a Tukey map from  $[\lambda]^{<\omega}$  to  $D(S) \times D(T)$  with the product ordering iff  $S \cap T$  is nonstationary.*

**Corollary 2.28** (Lemma 3, [37]). *If  $\lambda$  is a regular uncountable cardinal and  $S$  is an unbounded subset of  $E_\omega^\lambda$ , then there is a Tukey map from  $[\lambda]^{<\omega}$  to  $D(S)$  iff  $S$  is nonstationary.*

It follows that there is a Tukey map  $T : [\kappa]^{<\omega} \rightarrow D_0 \times D_1$ . Note that we can take such a Tukey map to have a cofinal range. Indeed, since  $\kappa^{\aleph_0} = \kappa$  we can fix a bijection  $f : [\kappa]^{<\omega} \rightarrow D_0 \times D_1$ . Now,  $D_0 \times D_1$  is directed, so the map  $S$  which



takes  $x \in [\kappa]^{<\omega}$  into a common upper bound of  $f(x)$  and  $T(x)$  is well-defined. It is easy to see that  $S$  is a Tukey map with a cofinal range.

**Example 2.29.** There are  $T_{3.5}$  spaces  $X$  and  $Y$  such that

$$\text{Nt}(X \times Y) < \min\{\text{Nt}(X), \text{Nt}(Y)\}.$$

*Proof.* For  $i = 0, 1$ , let  $X_i$  be the set  $[S_i]^{<\omega}$  topologized in such a way that a local base at the point  $x \in X_i$  is  $\{[x, E]_i : E \in D_i\}$ , where  $[x, E]_i = \{x \cup z : z \in [S_i \setminus E]^{<\omega}\}$ . Observe that the resulting topology on each  $X_i$  is  $T_1$  and that the above local bases are clopen local bases. Therefore, each  $X_i$  is  $T_{3.5}$ . We claim that we even have  $\chi\text{Nt}(X_i) \geq \aleph_1$  for  $i = 0, 1$ . Indeed, let  $\mathcal{B}$  be a local base at the point  $x \in X_i$ . Since  $\kappa^\omega = \kappa$ , we can assume that  $|\mathcal{B}| = \kappa$ . Moreover, we can assume that  $\mathcal{B}$  is of the form  $\{[x, E]_i : E \in \mathcal{E}\}$  where  $\mathcal{E} \subset D_i$  is cofinal. Now fix an injection  $F : [\kappa]^{<\omega} \rightarrow \mathcal{E}$ . By Corollary 2.28, we can find an unbounded set  $A$  such that  $\{F(a) : a \in A\}$  is bounded by some  $E$ . Therefore, we have  $[x, E]_i \subset [x, F(a)]_i$  for every  $a \in A$ , which shows that  $\chi\text{Nt}(X) \geq \aleph_1$ .

Now we claim that  $\text{Nt}(X_0 \times X_1) = \omega$ . Indeed, let  $T : [\kappa]^{<\omega} \rightarrow D_0 \times D_1$  be a Tukey map with cofinal range, and consider

$$\mathcal{A} = \{[x, T(y)_0]_0 \times [z, T(y)_1]_1 : x \in [S_0]^{<\omega}, y \in [\kappa]^{<\omega}, z \in [S_1]^{<\omega}\}.$$

This set is a base because the range of  $T$  is cofinal. Suppose that

$$[x, T(y)_0]_0 \times [x', T(y)_1]_1 \subset [x_j, T(y_j)_0]_0 \times [x'_j, T(y_j)_1]_1$$

for every  $j \in \omega$ . Then for every  $j \in \omega$  we have  $x_j \subset x$  and  $x'_j \subset x'$ . Hence,  $\{(x_j, x'_j) : j \in \omega\}$  is finite. Moreover,  $T(y_j)_0 \subset T(y)_0 \cup x$  and  $T(y_j)_1 \subset T(y)_1 \cup x'$  for all  $j \in \omega$ . Thus,  $\{T(y_j) : j \in \omega\}$  is bounded in  $D_0 \times D_1$ . Since  $T$  is a Tukey map,  $\{y_j : j \in \omega\}$  is bounded in  $[\kappa]^{<\omega}$ . Hence,  $\{(x_j, x'_j, y_j) : j \in \omega\}$  is finite. Thus,  $\mathcal{A}$  is  $\omega^{\text{op}}$ -like.  $\square$

*Question 2.30.* Do there exist compact spaces  $X$  and  $Y$  such that  $\text{Nt}(X \times Y) < \min\{\text{Nt}(X), \text{Nt}(Y)\}$ ?

The methods of this section can be used to attack also Question 2 from [6], which in our terminology reads *does every dense subspace of a regular space of countable Noetherian type have countable Noetherian type?* This is because of the following theorem.

**Theorem 2.31.** *Let  $X$  be a regular space such that every base of  $X$  contains a  $\text{Nt}(X)^{\text{op}}$ -like base of  $X$ . Then  $\text{Nt}(D) \leq \text{Nt}(X)$  for every dense  $D \subset X$ .*

*Proof.* Let  $\mathcal{B}$  be a base consisting of regular open sets (that is,  $\text{Int}(\overline{B}) = B$  for every  $B \in \mathcal{B}$ ). Let  $\mathcal{U} \subset \mathcal{B}$  be a  $\text{Nt}(X)^{\text{op}}$ -like. Let  $\mathcal{V} = \{D \cap U : B \in \mathcal{U}\}$ . Then  $\mathcal{V}$  is a base for  $D$ . To see that  $\mathcal{U}$  is  $\text{Nt}(X)^{\text{op}}$ -like just note that  $U \cap D \subset V \cap D$  implies that  $U \subset V$  whenever  $U$  and  $V$  are regular open.  $\square$

Define  $\delta\text{Nt}(X) = \sup\{\text{Nt}(D) : D \text{ is a dense subset of } X\}$ . Note that we always have  $\delta\text{Nt}(X) \geq \text{Nt}(X)$ . It is well-known that performing the same procedure for cellularity doesn't give rise to a new cardinal function. In other words, the cellularity of a dense subspace is always equal to the cellularity of the whole space. However, the authors of [6] showed that this is not the case for Noetherian type, at least if one is willing to forgo regularity.

**Theorem 2.32.** [6] *There is a Hausdorff space  $X$  such that  $\delta\text{Nt}(X) > \text{Nt}(X)$ .*

**Corollary 2.33.**  $\delta\text{Nt}(X) = \text{Nt}(X)$  whenever  $X$  is a compact space such that  $w(X)$  has regular weight and one of the following conditions holds:

- (1)  $X$  is homogeneous.
- (2)  $X$  is hereditarily normal.
- (3)  $\beta\omega$  does not embed in  $X$ .
- (4)  $|X| < 2^{w(X)}$ .

So Corollary 2.33 provides partial answers to Question 2 from [6], which we now pose in a more general form.

*Question 2.34.* Is  $\delta\text{Nt}(X) = \text{Nt}(X)$  for every regular space  $X$ ?

Bailey [4] introduced a natural strengthening of countable Noetherian type which implies countable Noetherian type of every dense subspace.

### 3. ON THE NOETHERIAN TYPE OF THE $G_\kappa$ -MODIFICATIONS OF A COMPACT SPACE

Let  $X_\theta$  denote the space obtained from  $X$  by declaring all  $< \theta$ -sized intersections of open sets open. We denote the space  $X_{\aleph_1}$  also by the symbol  $X_\delta$ , since it coincides with the topology generated by all  $G_\delta$  subsets of  $X$ . This is also called the  $G_\delta$ -modification of  $X$ , and has been extensively studied in the literature. A natural problem in this area is to find a bound for a given cardinal invariant on  $X_\delta$  in terms of its value on  $X$ . For example, in [19], Juhász solved this problem for the cellularity with the following elegant inequality.

**Theorem 3.1.** (*Juhász*) *Let  $X$  be compact. Then  $c(X_\delta) \leq 2^{c(X)}$ .*

We solve this problem for Noetherian type in Theorem 3.2. To show that our theorem is the sharpest possible, we look at the Noetherian type of the space  $(2^\kappa)_\theta$ . This pursuit is interesting on its own due to connections with PCF theory and the occurrence of independence phenomena. In fact, while the Noetherian type of  $(2^\kappa)_\theta$  for  $\theta = \aleph_0$  (i.e.,  $\text{Nt}(2^\kappa)$ ) is  $\aleph_0$  in ZFC, for  $\theta > \aleph_0$  it is not as easy to determine. We are able to show that its value for  $\theta > \aleph_0$  and  $\kappa = \aleph_\omega$ , the least singular cardinal, cannot be determined in ZFC modulo the consistency of certain very large cardinals, but is bounded in ZFC by  $\aleph_4$  whenever  $\theta < \aleph_4$ .

We begin with the promised upper bound on the Noetherian type of  $X_\delta$  for compact  $X$  from a cardinal arithmetic assumption:

**Theorem 3.2.** *Suppose that  $\lambda < \kappa \Rightarrow \lambda^{\aleph_0} \leq \kappa$  for every cardinal  $\kappa$ . Every countably compact space  $X$  such that the cofinality of  $\text{Nt}(X)$  is uncountable then satisfies  $\text{Nt}(X_\delta) \leq \text{Nt}(X)^+$ .*

The theorem is an immediate consequence of the following lemma.

**Lemma 3.3.** *Suppose that  $X$  is a countably compact regular space,  $\kappa$  is a cardinal of uncountable cofinality, and  $\lambda < \kappa \Rightarrow \lambda^{\aleph_0} < \kappa$ . Then  $\text{Nt}(X) \leq \kappa \Rightarrow \text{Nt}(X_\delta) \leq \kappa$ .*

*Proof.* Let  $\mathcal{B}$  be a  $\kappa^{\text{op}}$ -like base for  $X$ . Moreover, let  $\mathcal{B}_\delta$  be the set of all countable intersections from  $\mathcal{B}$ . Clearly  $\mathcal{B}_\delta$  is a base for  $X_\delta$ . Now, suppose it's not  $\kappa^{\text{op}}$ -like. Then some  $B \in \mathcal{B}_\delta$  is contained in every element of some family  $\mathcal{F} = \{B_\alpha : \alpha < \kappa\} \subset \mathcal{B}_\delta$  of distinct  $G_\delta$  sets. Let  $\mathcal{U} \subset \mathcal{B}$  be the set of all open sets that make up elements in  $\mathcal{F}$ . Then  $|\mathcal{U}| \geq \kappa$ , because if  $|\mathcal{U}| < \kappa$  then  $|\mathcal{F}| \leq |\mathcal{U}|^{\aleph_0} < \kappa$ . So take some enumeration  $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ . Observe that every nonempty  $G_\delta$  set in a regular

space contains a nonempty closed  $G_\delta$ , then let  $G = \bigcap_{i \in \omega} \overline{G_i} \subset B$  be some closed  $G_\delta$  set. Observe that  $G \subset U_\alpha$  for every  $\alpha < \kappa$ , so use countable compactness to find for every  $\alpha < \kappa$  an  $n \in \omega$  such that  $\bigcap_{i < n} G_i \subset U_\alpha$ . Since  $\kappa$  has uncountable cofinality there has to be some  $R \subset \kappa$  and  $n \in \omega$  such that  $|R| = \kappa$  and  $\bigcap_{i < n} G_i \subset U_\alpha$  for every  $\alpha \in R$ . Let now  $V \in \mathcal{B}$  be such that  $V \subset \bigcap_{i=1}^n G_i$ . Then  $\kappa^{\text{op}}$ -ness of  $\mathcal{B}$  is contradicted.  $\square$

Judy Roitman asked if countable compactness is essential in the above theorem. Our next example shows that it is.

**Example 3.4.** Suppose  $\kappa$  is a regular cardinal such that  $\kappa^\omega = \kappa > \aleph_1$ . There then is a  $T_{3.5}$  space  $Z$  such that  $\text{Nt}(Z) = \aleph_1$  and  $\text{Nt}(Z_\delta) = \kappa^+$ .

*Proof.* Let  $S = \{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ ,  $X = [S]^{<\omega}$ , and  $Y = \{\alpha < \kappa : \text{cf}(\alpha) \geq \omega_1\}$  with the subspace topology from  $\kappa$ ; let  $\mathcal{D}$  be the set of countable compact subsets of  $S$ ; let  $Z = Y \cup (X \times \omega)$  with the topology generated by set  $\mathcal{B}$  of sets of the form  $[x, E] \times \{n\}$  where  $x \in X$ ,  $E \in \mathcal{D}$ , and  $n \in \omega$ , and sets of the form  $(J \cap Y) \cup ([x, E] \times (\omega \setminus n))$  where  $J$  is a nonempty open subinterval of  $\kappa$ ,  $x \in X$ ,  $E \in \mathcal{D}$ ,  $n \in \omega$ , and  $x < J < E$  in the sense below.

$$\alpha < \beta \text{ for all } \langle \alpha, \beta \rangle \in (x \times J) \cup (J \times E)$$

Like in Example 2.29,  $[x, E]$  denotes  $\{x \cup z : z \in [S \setminus E]^{<\omega}\}$ . Observe that if  $(J_i \cap Y) \cup ([x_i, E_i] \times (\omega \setminus n_i)) \in \mathcal{B}$  for all  $i < 2$ , and  $J_0 \cap J_1 \neq \emptyset$ , then

$$(J_0 \cap J_1 \cap Y) \cup ([x_0 \cup x_1, E_0 \cup E_1] \times (\omega \setminus (n_0 \cup n_1))) \in \mathcal{B}.$$

It follows that  $\mathcal{B}$  is a base of  $Z$ . Hence,  $X \times \omega$  is dense open in  $Z$ . Also observe that for every  $B \in \mathcal{B}$  and  $p \in Z \setminus B$ , there exists  $A \in \mathcal{B}$  such that  $p \in A$  and  $A \cap B = \emptyset$ . Hence,  $\mathcal{B}$  is a clopen base. It is also easy to check that  $Z$  is  $T_1$ . Therefore,  $Z$  is  $T_{3.5}$ .

Just as in the proof of Example 2.29, Corollary 2.28 implies that  $\chi \text{Nt}(X \times \{0\}) \geq \aleph_1$ . Since  $X \times \{0\}$  is an open subspace of  $Z$ , must have  $\text{Nt}(Z) \geq \aleph_1$ . Let us show that also  $\text{Nt}(Z) \leq \aleph_1$ . Let  $\mathcal{U} = \{U_\alpha : \alpha \in S\}$  be a base of  $Z$ . Let  $\mathcal{V} = \{U_\alpha \cap V_{\alpha, n} : \alpha \in S \wedge n < \omega\} \setminus \{\emptyset\}$  where

$$V_{\alpha, 0} = ((\alpha, \kappa) \cap Y) \cup (\{\alpha\} \times \omega)$$

and, for all  $n > 0$ ,

$$V_{\alpha, n} = ([0, \alpha + 1) \cap Y) \cup (\{\emptyset, \{\alpha + \omega \cdot n\}\} \times \omega).$$

Since  $\bigcup_{n < \omega} V_{\alpha, n} = Z$  for all  $\alpha$ , the set  $\mathcal{V}$  is a base of  $Z$ . If  $\mathcal{A} \in [\mathcal{V}]^{\aleph_1}$ , then there exist  $I \in [S]^{\aleph_1}$  and  $n < \omega$  such that  $U_i \cap V_{i, n} \in \mathcal{A}$  for all  $i \in I$ . It is easily checked that  $\bigcap_{i \in I} V_{i, n} \setminus Y$  has empty interior in  $X \times \omega$ . Since  $X \times \omega$  is dense open in  $Z$ , the set  $\bigcap_{i \in I} V_{i, n}$  has empty interior in  $Z$ . Therefore,  $\mathcal{V}$  is  $\aleph_1^{\text{op}}$ -like.

Finally, let us show that  $\text{Nt}(Z_\delta) = \kappa^+$ . The topology of  $Y_\delta$  and the subspace topologies  $Y$  inherits from  $Z_\delta$  and  $\kappa_\delta$  are all identical. Let  $T = \{y \in Y : \text{cf}(Y \cap y) \geq \omega_1\}$ , which is stationary in  $\kappa$ . For every  $t \in T$  and every  $Y_\delta$ -neighborhood of  $U$  of  $t$ , the point  $\min(U)$  is necessarily less than  $t$  and isolated in  $Y_\delta$ . Hence, by the Pressing Down Lemma, if  $\mathcal{W}$  is a base of  $Y_\delta$ , then there must exist  $\kappa$ -many distinct elements  $\langle W_i : i < \kappa \rangle$  of  $\mathcal{W}$  with the same isolated minimum  $\beta$ , which implies that  $\beta$  is in the interior of  $\bigcap_{i < \kappa} W_i$ . Thus,  $\text{Nt}(Y_\delta) = \kappa^+$ . Since each  $X \times \{n\}$  is closed in  $Z$ , the set  $Y$  is open in  $Z_\delta$ . Therefore,  $\text{Nt}(Z_\delta) = \kappa^+$  too.  $\square$

As the following example shows, the cardinal arithmetic assumption in Theorem 3.2 is also essential, even if we weaken the conclusion to  $\text{Nt}(X_\delta) \leq 2^{\text{Nt}(X)}$ .

**Definition 3.5.**  $\text{cov}(\theta, \kappa)$  is the least size of a collection  $\mathcal{A} \subseteq [\theta]^\kappa$  such that every  $X \in [\theta]^\kappa$  is contained in some member of the collection.

**Example 3.6.** If  $2^{<\aleph_\omega} = \aleph_\omega$  and  $2^{\aleph_\omega} \geq 2^{\aleph_{\omega+1}}$ , then there is a compact space  $X$  such that  $\text{cf}(\text{Nt}(X)) > \aleph_0$  with  $\text{Nt}(X_\delta) > 2^{\text{Nt}(X)}$ .

Note that Gitik-Magidor forcing ([16], see also [15]) produces a model with the above cardinal arithmetic, assuming ZFC+GCH+“ $\kappa$  is a measurable cardinal of Mitchell order  $\kappa^{++}$ ”.

*Proof.* Assuming the above cardinal arithmetic, we must have  $\text{cov}(\aleph_\omega, \aleph_0) \geq 2^{\aleph_{\omega+1}}$  because  $2^{\aleph_\omega} = (2^{<\aleph_\omega})^{\aleph_0}$  and  $(\aleph_\omega)^{\aleph_0} = \mathfrak{c} \cdot \text{cov}(\aleph_\omega, \aleph_0)$ . Let  $X$  be the one-point compactification of  $\aleph_\omega$  with the discrete topology. Then  $\text{Nt}(X) = \aleph_{\omega+1}$ . (See Example 2.18.) We show now that  $\text{Nt}(X_\delta) = \text{cov}(\aleph_\omega, \aleph_0)^+$  so that  $X$  will satisfy the conclusion of the example. Indeed, note that  $\text{Nt}(X_\delta) \leq \text{cov}(\aleph_\omega, \aleph_0)^+$  since  $w(X_\delta) = \text{cov}(\aleph_\omega, \aleph_0)$  and  $\text{Nt}(X_\delta) \leq w(X_\delta)^+$ . For the reverse inequality, let  $\lambda = \text{cov}(\aleph_\omega, \aleph_0)$  and  $\mathcal{B}$  be any base for  $X$ , and suppose by contradiction that  $\text{Nt}(X) \leq \lambda$ . Let  $\mathcal{C} = \{C \in [\aleph_\omega]^{\aleph_0} : X \setminus C \in \mathcal{B}\}$ . Enumerate  $\mathcal{C} = \{C_\alpha : \alpha < \lambda\}$ . Let  $\gamma$  be any ordinal less than  $\aleph_\omega$ . If we could find  $\lambda$ -many elements of  $\mathcal{C}$  which miss  $\gamma$ , then the isolated point  $\gamma$  would have  $\lambda$ -many supersets in  $\mathcal{B}$ . Hence, using the regularity of  $\lambda$  (see Theorem 3.17), we can assume that for every  $\alpha < \aleph_\omega$ , we can find  $\beta_\alpha < \lambda$  such that  $\alpha \in C_\gamma$  for every  $\gamma \geq \beta_\alpha$ . Let  $\beta = \sup_{\alpha < \omega_1} \beta_\alpha$ . We have that  $\beta < \lambda$ , again by regularity of  $\lambda$ . But this implies  $\omega_1 \subseteq C_{\beta+1}$ , which contradicts the fact that  $C_{\beta+1}$  is countable. Therefore,  $\text{Nt}(X) \geq \lambda^+$  and we are done.  $\square$

By Lemma 3.3,  $\text{Nt}(X_\delta) \leq (\text{Nt}(X)^\omega)^+$  for all compact  $X$ ; in particular,  $\text{Nt}((2^{\aleph_\omega})_\delta) \leq \mathfrak{c}^+$ . However, Theorem 3.25 below shows, modulo very large cardinals, that the upper bound  $\text{Nt}((2^{\aleph_\omega})_\delta) \leq \mathfrak{c}^+$  cannot be improved. Thus, the assumption  $\text{cf}(\text{Nt}(X_\delta)) > \aleph_0$  is also essential to Theorem 3.2, even if we weaken the conclusion to  $\text{Nt}(X_\delta) \leq 2^{\text{Nt}(X)}$ .

### 3.1. The Noetherian type of box products of Cantor cubes.

3.1.1. *Sparse families.* We now introduce the main combinatorial object of the rest of the paper.

#### Definition 3.7.

- (1) Let  $\kappa$  be a cardinal. A family of sets  $\mathcal{F}$  is  $\kappa$ -small if  $|\bigcup \mathcal{F}| < \kappa$ . Equivalently, there exists a set  $B$  with  $|B| < \kappa$  such that  $\mathcal{F} \subseteq \mathcal{P}(B)$ .
- (2) A family of sets  $\mathcal{F}$  is  $(\mu, \kappa)$ -sparse if no  $\mathcal{G} \subset \mathcal{F}$  with  $|\mathcal{G}| \geq \mu$  is  $\kappa$ -small. In other words,  $|\bigcup \mathcal{G}| \geq \kappa$  for every  $\mathcal{G} \in [\mathcal{F}]^\mu$ .
- (3) A family  $\mathcal{F}$  is called  $\nu$ -uniform if each member of  $\mathcal{F}$  is a set of cardinality  $\nu$ .

Let us list a few basic properties of  $(\mu, \kappa)$ -sparse families of sets.

#### Proposition 3.8.

- (1) A  $(\mu, \kappa)$ -sparse family  $\mathcal{F}$  is  $(\mu', \kappa')$ -sparse whenever  $\mu' \geq \mu$  and  $\kappa' \leq \kappa$ .
- (2) Every  $\nu$ -uniform  $\mathcal{F}$  is  $((2^\nu)^+, \nu^+)$ -sparse.

- (3) If  $\mu > |\mathcal{F}|$  then  $\mathcal{F}$  is  $(\mu, \kappa)$ -sparse for every cardinal  $\kappa$  (vacuously) and if  $\kappa > |\bigcup \mathcal{F}|$  then  $\mathcal{F}$  is not  $(\mu, \kappa)$ -sparse for any  $\mu$ .
- (4) If  $\kappa$  is limit and  $\mathcal{F}$  is  $(\mu, \theta)$ -sparse for every  $\theta < \kappa$  then  $\mathcal{F}$  is  $(\mu, \kappa)$ -sparse.
- (5) For every cardinal  $\kappa$  the class of cardinals  $\mu$  for which  $\mathcal{F}$  is not  $(\mu, \kappa)$ -sparse is closed under limits of cofinality  $< \text{cf}(\kappa)$ .
- (6) If  $\mathcal{F}$  is  $\nu$ -uniform then the least  $\mu$  for which  $\mathcal{F}$  is  $(\mu, \nu^+)$ -sparse satisfies  $\text{cf}(\mu) > \nu$ .
- (7) If the relation  $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_{n+1}, \aleph_n)$  holds, then every  $(\aleph_{n+1}, \aleph_{n+1})$ -sparse  $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$  has cardinality at most  $\aleph_\omega$ .

*Proof.* The first 4 items are obvious and (6) follows from (5). To prove (5) suppose  $\langle \mu_i : i < \theta \rangle$  is an increasing sequence of ordinals with limit  $\mu$  for some  $\theta < \text{cf}(\kappa)$  and that  $\mathcal{F}$  is not  $(\mu_i, \kappa)$ -sparse for each  $i < \theta$ . For each  $i < \theta$  fix a  $\kappa$ -small  $\mathcal{G}_i \subseteq \mathcal{F}$  of cardinality  $\mu_i$  and let  $\mathcal{G} = \bigcup_{i < \theta} \mathcal{G}_i$ . Now  $\mathcal{G}$  has cardinality  $\mu$  and is  $\kappa$ -small because  $\theta < \text{cf}(\kappa)$ .

Recall that the symbol  $(\kappa, \lambda) \rightarrow (\alpha, \beta)$  stands for the statement that for every structure  $M = (A, B, \dots)$  with countable signature,  $|A| = \kappa$ , and  $|B| = \lambda$ , there is an elementary substructure  $N = (C, D, \dots) \prec M$  such that  $|C| = \alpha$  and  $|D| = \beta$ . If  $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$  has cardinality  $\aleph_{\omega+1}$  and is  $(\aleph_{n+1}, \aleph_{n+1})$  sparse, and  $M \prec (H(\Omega), \aleph_\omega, \mathcal{F}, \dots)$  is an elementary submodel of cardinality  $\aleph_{\omega+1}$  with  $C^M = \aleph_\omega$  and  $\mathcal{F} \subseteq M$ , then every elementary submodel  $N \prec M$  for which  $\mathcal{F}^N$  has cardinality  $\aleph_{n+1}$  must have  $|C^N| = |N \cap \aleph_\omega| \geq \aleph_{n+1}$  as well, since  $A \in \mathcal{F} \cap N$  implies that  $A \subseteq C^N$  and  $\mathcal{F}$  is  $(\aleph_{n+1}, \aleph_{n+1})$ -sparse. Thus (7) follows.  $\square$

**Proposition 3.9.** *Suppose  $\mathcal{F}$  is  $\nu$ -uniform and  $(\mu, \nu^+)$ -sparse. Then  $\mathcal{F}$  is  $(\mu, \kappa)$ -sparse for every  $\kappa \geq \nu^+$  such that for all  $\nu < \rho < \kappa$  it holds that  $\text{cov}(\rho, \nu) < \text{cf}(\mu)$ .*

*Proof.* Suppose that, contrary to the claim, there exists a set  $B$ , with  $|B| = \rho < \kappa$  and  $|\mathcal{P}(B) \cap \mathcal{F}| \geq \mu$ . Fix a covering collection  $\mathcal{B} \subseteq [B]^{<\nu}$  of cardinality  $|\mathcal{B}| < \text{cf}(\mu)$ . It follows that some  $Y \in \mathcal{B}$  contains  $\mu$  members of  $\mathcal{F}$  which, as  $|Y| = \nu$ , contradicts  $(\mu, \nu^+)$ -sparseness.  $\square$

**Proposition 3.10.** *Suppose that  $\mathcal{F}$  is  $\aleph_\alpha$ -uniform for some infinite cardinal  $\aleph_\alpha$ , and  $\mu$  is the least cardinal for which  $\mathcal{F}$  is  $(\mu, \aleph_{\alpha+1})$ -sparse. Then  $\mathcal{F}$  is  $(\mu', \aleph_{\alpha+\beta})$ -sparse for every  $1 \leq \beta \leq \omega$  and  $\mu' = \max\{\mu, \aleph_{\alpha+\beta}\}$ .*

*Proof.* The case  $\beta = \omega$  follows from the case  $1 \leq \beta < \omega$ , which we prove by induction on  $n$ .

Assume that  $\mathcal{F}$  is  $(\mu, \aleph_{\alpha+n})$ -sparse. By Proposition 3.8, (6) we have  $\text{cf}(\mu) > \aleph_{\alpha+n}$ . Let now  $\mu' = \max\{\mu, \aleph_{\alpha+n+1}\}$  and note that we also have  $\text{cf}(\mu') > \aleph_{\alpha+n}$ . For all  $\rho$  such that  $\aleph_\alpha < \rho < \aleph_{\alpha+n+1}$ , we have  $\text{cov}(\rho, \aleph_\alpha) = \rho$ , so, by Proposition 3.9,  $\mathcal{F}$  is  $(\mu', \aleph_{\alpha+n+1})$ -sparse.  $\square$

**Corollary 3.11.** *If  $\mathcal{F}$  is an  $\aleph_0$ -uniform family and there exists  $n$  such that  $\mathcal{F}$  is  $(\aleph_n, \aleph_1)$ -sparse then  $\mathcal{F}$  is  $(\aleph_\alpha, \aleph_\alpha)$ -sparse for all  $n \leq \alpha \leq \omega$ .*

We also note the following easily proved proposition.

**Proposition 3.12.** *Let  $\lambda = \text{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ . If  $\{F_\alpha : \alpha < \lambda\} \subseteq [\aleph_\omega]^{\aleph_0}$  is a  $(\mu, \kappa)$ -sparse family and  $\{G_\alpha : \alpha < \lambda\}$  is any family which is cofinal in  $([\aleph_\omega]^{\aleph_0}, \subseteq)$  then  $\{F_\alpha \cup G_\alpha : \alpha < \lambda\}$  is both  $(\mu, \kappa)$ -sparse and cofinal.*

Sparse families generalize Shelah's  $\kappa$ -free families studied for example in [31] and in Magidor and Shelah's [24]. A family of sets is  $\kappa$ -free if each of its  $\kappa$ -sized subfamilies has an injective choice function; this condition of course implies  $(\kappa, \kappa)$ -sparseness. The converse is not true, but we have the following weaker implication:

**Proposition 3.13.** *Let  $\mathcal{F}$  be a  $(\kappa, \kappa)$ -sparse family of countable sets and  $\mathcal{A} \subset \mathcal{F}$  be any subfamily of size  $\kappa$ . Then  $\mathcal{A}$  contains a  $\kappa$ -free family of size  $\kappa$ .*

*Proof.* By sparseness,  $\bigcup \mathcal{A}$  has size  $\kappa$ , so let  $\{x_\alpha : \alpha < \kappa\}$  be an enumeration of it. Suppose that, for some  $\beta < \kappa$  you have constructed  $\{F_\alpha : \alpha < \beta\} \subset \mathcal{A}$  and ordinals  $\{\gamma_\alpha : \alpha < \beta\}$  such that  $x_{\gamma_\alpha} \in F_\alpha$ . There then is an ordinal  $\tau$  such that  $x_\tau \notin \bigcup_{\alpha < \beta} F_{\gamma_\alpha}$ . So, let  $\gamma_\beta := \tau$  and choose  $S \in \mathcal{A}$  such that  $x_{\gamma_\beta} \in S$ . Let  $F_\beta := S$ . At the end of the induction,  $\{F_\beta : \beta < \kappa\}$  is a free subfamily of  $\mathcal{A}$ .  $\square$

The following theorem links sparse cofinal families and the Noetherian type of box product topologies on Cantor Cubes. Clearly, for every  $\kappa < \aleph_\omega$  the topology  $(2^{\aleph_\omega})_{\kappa+}$  is the box topology with boxes of cardinality  $\leq \kappa$ .

**Theorem 3.14.** *Let  $\kappa, \theta$  and  $\lambda$  be cardinals with  $\aleph_0 \leq \theta \leq \kappa$  and  $\theta \leq \lambda$ . Let  $Y \subset (2^\lambda)_\theta$  be a dense subset. Then  $\text{Nt}(Y) \leq \kappa$  if and only if there is a  $(\kappa, \theta)$ -sparse cofinal family in  $([\lambda]^{<\theta}, \subseteq)$ .*

*Proof.* Let  $\mathcal{F}$  be a  $(\kappa, \theta)$ -sparse cofinal family. Let  $\mathcal{B} = \{[\sigma] \cap Y : \text{dom } \sigma \in \mathcal{F}\}$ . It is easy to see that  $\mathcal{B}$  is a base for  $Y$ . To see that it is  $\kappa^{op}$ -like suppose by contradiction that there is a  $< \theta$ -sized partial function  $[\sigma]$  and a family of  $< \theta$ -sized partial functions  $\{\sigma_\alpha : \alpha < \kappa\}$  such that  $[\sigma] \cap Y \subset [\sigma_\alpha] \cap Y$  for every  $\alpha < \kappa$  and  $[\sigma_\alpha] \neq [\sigma_\beta]$  whenever  $\alpha \neq \beta$ . By taking closures we see that  $[\sigma] \subset [\sigma_\alpha]$  for every  $\alpha < \kappa$ . Note that when  $\alpha \neq \beta$ ,  $\text{dom } \sigma_\alpha$  and  $\text{dom } \sigma_\beta$  are distinct or otherwise the corresponding basic open sets would be disjoint. Now  $\text{dom } \sigma_\alpha \subset \text{dom } \sigma$  for every  $\alpha < \kappa$ , which contradicts  $(\kappa, \theta)$ -sparseness of the family  $\mathcal{F}$ .

Viceversa, suppose that  $\text{Nt}(Y) \leq \kappa$  and let  $x \in Y$ . Let  $\mathcal{B}$  be a  $\kappa^{op}$ -like local base at  $x$ . For every  $B \in \mathcal{B}$  let  $\sigma_B$  be a  $< \theta$ -sized partial function such that  $x \in [\sigma_B] \cap Y \subset B$  and let  $\mathcal{B}' = \{[\sigma_B] : B \in \mathcal{B}\}$ . Since  $\mathcal{B}$  is  $\kappa^{op}$ -like and  $Y$  is dense,  $\mathcal{B}'$  is a  $\kappa^{op}$  like local base at  $x$  in  $(2^\lambda)_\theta$ . Hence  $\{\text{dom}(\sigma) : [\sigma] \in \mathcal{B}'\}$  is a  $(\kappa, \theta)$ -sparse cofinal family. Indeed, suppose by contradiction that there is a family of distinct partial functions  $\{\sigma_\alpha : \alpha < \kappa\}$  such that  $[\sigma_\alpha] \in \mathcal{B}'$  and  $|\bigcup_{\alpha < \kappa} \text{dom } \sigma_\alpha| < \theta$ . Note that, since  $x \in [\sigma_\alpha]$  for every  $\alpha < \kappa$ , then  $\sigma_\alpha$  and  $\sigma_\beta$  are compatible for every  $\alpha \neq \beta$ . So  $\tau := \bigcup_{\alpha < \kappa} \sigma_\alpha$  is a  $< \theta$ -sized partial function such that  $[\tau] \subset [\sigma_\alpha]$  for every  $\alpha < \kappa$ , which contradicts the fact that  $\mathcal{B}'$  is  $\kappa^{op}$ -like.  $\square$

**Corollary 3.15.** *Let  $n$  and  $m$  be positive integers.*

$$\text{Nt}((2^{\aleph_n})_{\aleph_m}) = \begin{cases} \aleph_0 & \text{if } m > n \\ \aleph_{m+1} & \text{if } m \leq n \end{cases}$$

*Proof.* The first case holds as  $m > n$  implies that  $(2^{\aleph_n})_{\aleph_m}$  is discrete. Assume  $m - 1 < n$ . Simple induction shows that  $\text{cov}(\aleph_n, \aleph_{m-1}) = \aleph_n$ . Let  $\{F_\alpha : \alpha < \aleph_n\}$  enumerate a cofinal subset of  $([\aleph_n]^{\aleph_{m-1}}, \subseteq)$ . The family  $\{F_\alpha \cup \{\alpha\} : \alpha < \aleph_n\}$  is  $(\aleph_m, \aleph_m)$ -sparse and cofinal family in  $([\aleph_n]^{\aleph_{m-1}}, \subseteq)$ . Hence  $\text{Nt}((2^{\aleph_n})_{\aleph_m}) \leq \aleph_m$ . Since there is no  $(\aleph_{m-1}, \aleph_m)$ -sparse family in  $([\aleph_n]^{\aleph_{m-1}}, \subseteq)$ , we have  $\text{Nt}((2^{\aleph_n})_{\aleph_m}) > \aleph_{m-1}$  and the second case is done.  $\square$

**Example 3.16.** In Theorem 3.14, we cannot weaken density of  $Y$  to somewhere density. To see that, note that if  $A$  denotes the one-point compactification of a discrete set of size  $\aleph_\omega$  then  $A_\delta$  embeds homeomorphically into  $X = (2^{\aleph_\omega})_\delta$  (simply, let  $f_\alpha \in X$  be the function defined by  $f_\alpha(\beta) = 1$  if and only if  $\alpha = \beta$  and let  $f$  be the null function. Then  $\{f_\alpha : \alpha < \aleph_\omega\} \cup \{f\}$  is homeomorphic to  $Y$ ). Let  $U$  and  $V$  be disjoint basic open sets of  $X$ . Then  $U$  and  $V$  are homeomorphic to  $X$  and hence  $V$  contains a copy  $Z$  of  $A_\delta$ . By the proof of Example 3.6  $U \oplus Z$  is a space of Noetherian type  $\text{cov}(\aleph_\omega, \omega)^+$  embedding homeomorphically into  $X$  as a somewhere dense set, while in Theorem 3.20 we shall show that there is an  $(\aleph_4, \aleph_1)$ -sparse cofinal family using only the axioms of ZFC.

**3.2. Sparse families from PCF theory.** In this Section we show that a cofinal  $(\aleph_4, \aleph_1)$ -sparse  $\mathcal{F} \subseteq [\aleph_\omega]$  exists, and thus bound  $\text{Nt}((2^{\aleph_\omega})_\delta)$  in ZFC.

The existence of such a family follows from the fact that all points of cofinality  $\aleph_n$  for  $n \geq 4$  in a sufficiently thin PCF scale are flat. The above existence of many flat points is mentioned in footnote 5 in [30], follows from Lemma 2.12, 2.19 in [1], and is presented also in the forthcoming [34], in which Shelah handles reflection properties of sets related to the ideal  $I[\lambda]$ .

We give in 3.20 below a direct proof that the family of ranges of all members of a sufficiently thin maximal PCF scale form an  $(\aleph_4, \aleph_1)$ -sparse family. The proof is modeled after Shelah's spectacular proof [33] of the existence of a stationary set of ordinals of cofinality  $\kappa$  in the ideal  $I[\lambda]$  when  $\kappa^+ < \lambda$ .

**3.2.1. Background from PCF Theory.** To gain more insight on the order theory of bases in the countably supported box product topology, we need some concepts from PCF theory, which we now review for the reader's convenience.

The proofs of Shelah's theorems quoted below can be found in [31] (see also [1, 21] for expositions).

PCF theory studies the *possible* cofinalities of products of small sets of regular cardinals modulo filters. We recall the basic definitions for the particular case  $A = \{\aleph_n : n < \omega\}$ .

Let  $U$  be a filter on  $\omega$ . The relations  $=_U, \leq_U$  and  $<_U$  are defined on the set  $\prod_n \aleph_n/U$  in the obvious way, e.g.  $f <_U g \iff \{n : f(n) < g(n)\} \in U$ . The relations  $=, \leq, <$  without a filter subscript denote the pointwise relations.

The *bounding number*  $\mathfrak{b}(\prod_n \aleph_n/U)$  is the least cardinality of an unbounded subset of  $\prod_n \aleph_n/U$  and it always regular when  $U$  is a proper filter. If  $\mathfrak{b}(\prod_n \aleph_n/U) = \text{cf}(\prod_n \aleph_n/U)$  then  $\prod_n \aleph_n/U$  is said to have *true cofinality* (denoted by  $\text{tcf}$ ) and one can find a linearly ordered cofinal subset of it. Such a subset is called a *scale*.

Let  $\text{pcf}(A) = \{\text{tcf}(\prod A/U) : U \text{ is a filter on } A\}$ . An important theorem of PCF theory states that this set has a maximum.

**Theorem 3.17.** (*Shelah*) *If  $A = \{\aleph_n : n \in \omega\}$ , then  $\text{pcf}(A)$  is a set of regular cardinals with a maximum and  $\max \text{pcf}(A) = \text{cov}(\aleph_\omega, \aleph_0)$ .*

It is easy to realize that  $(\aleph_\omega)^\omega = \text{cov}(\aleph_\omega, \aleph_0) \cdot \mathfrak{c}$ . While the continuum has no bound in ZFC, PCF theory has produced a bound for  $\text{cov}(\aleph_\omega, \aleph_0)$ .

**Theorem 3.18.** (*Shelah*)  *$\text{cov}(\aleph_\omega, \aleph_0) < \aleph_{\omega_4}$ .*

The notion of a PCF scale will allow us to give a ZFC upper bound on the Noetherian type of the countably supported topology in Corollary 3.21. To prove

that the upper bound can consistently drop to  $\aleph_1$ , we will need PCF scales with stronger properties whose existence is independent of ZFC.

Recall that, given a filter  $U$  over  $\omega$ , a function  $g \in On^\omega$  is said to be an *exact upper bound* for a  $<_U$ -increasing sequence  $\{f_\alpha : \alpha < \lambda\} \subset On^\omega$  if  $f_\alpha <_U g$  for every  $\alpha < \lambda$  and whenever  $g' <_U g$  there is  $\beta < \lambda$  such that  $g' <_U f_\beta$ . A  $<_U$ -increasing sequence  $\{f_\alpha : \alpha < \beta\}$  where  $cf(\beta) = \delta > \aleph_0$  is called *flat* (see [20]) if there exists a  $<$ -increasing sequence  $\{h_i : i < \delta\} \subset On^\omega$  so that for all  $i < \delta$  there is  $\alpha < \beta$  with  $h_i <_U f_\alpha$  and for every  $\alpha < \beta$  there is  $i < \delta$  with  $f_\alpha <_U h_i$ . Observe that every flat sequence has an exact upper bound. By Lemma 9 in [20], if  $\lambda > \aleph_0$  is regular,  $U$  is a filter over  $\omega$  and a  $<_U$ -increasing sequence  $\langle f_\alpha : \alpha < \lambda \rangle$  of ordinal functions on  $\omega$  has an exact upper bound  $g$ , then for every regular  $\kappa \in (\aleph_0, \lambda)$  the set  $A_\kappa = \{n : cf(g(n)) = \kappa\}$  is in the dual of  $U$ , that is,  $\omega \setminus A_\kappa \in U$ . We will use the following simple corollary of this fact.

**Proposition 3.19.** *If  $\kappa > \aleph_0$  is regular,  $U$  is a filter over  $\omega$ , and a  $<_U$ -increasing sequence of ordinal functions  $\langle f_\alpha : \alpha < \kappa \rangle$  on  $\omega$  is flat, then  $\bigcup_{\alpha < \kappa} \text{ran } f_\alpha$  has cardinality  $\kappa$ .*

A scale  $\bar{f} = \{f_\alpha : \alpha < \lambda\} \subset On^\omega$  is called *good* if, for every  $\beta < \lambda$  such that  $cf(\beta) > \aleph_0$ , the sequence  $\bar{f} \upharpoonright \beta = \{f_\alpha : \alpha < \beta\}$  is flat and the function  $f_\beta$  is an exact upper bound for it.

3.2.2. *A ZFC upper bound.* We prove now the existence of a cofinal sparse family.

**Theorem 3.20.** *There exists a cofinal family  $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$  which is  $(\aleph_\alpha, \aleph_\alpha)$ -sparse for every  $4 \leq \alpha \leq \omega$ .*

**Corollary 3.21.** (1) *The Noetherian type of  $(2^{\aleph_\omega})_\delta$  is at most  $\aleph_4$ .*  
 (2) *For every  $n \in \omega$  the Noetherian type of  $(2^{\aleph_\omega})_{\aleph_n}$  is at most  $\max\{\aleph_4, \aleph_{n+1}\}$*   
 (3) *For every  $n \geq 4$ , the Noetherian type of  $(2^{\aleph_\omega})_{\aleph_n}$  is equal to  $\aleph_{n+1}$ .*  
 (4)  $(\aleph_{\omega+1}, \aleph_\omega) \not\approx (\aleph_{n+1}, \aleph_n)$  for all  $n \geq 3$ .

*Proof of Theorem.* By Corollary 3.11 and Proposition 3.12, it suffices to prove the existence of a family  $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$  of cardinality  $\text{cov}(\aleph_\omega, \aleph_0)$  which is  $(\aleph_4, \aleph_1)$ -sparse. The proof below makes no assumption about the size of the continuum, but, by Proposition 3.8, (2), the proof below is needed only when the continuum is larger than  $\aleph_3$ .

Let  $\Omega$  be a sufficiently large regular cardinal and let  $\langle H(\Omega), \in, \dots \rangle$  be the structure of all sets of hereditary cardinality smaller than  $\Omega$  expanded with Skolem functions. An object  $f(\bar{p})$  which a Skolem function  $f$  selects from nonempty set definable from the parameters  $\bar{p}$  will be called “canonical.” For example, for every regular  $\kappa, \lambda < \Omega$  that satisfy  $\kappa^+ < \lambda$ , there exists club guessing sequences of the form  $\bar{C} = \langle c_\delta : \delta \in E_\kappa^\lambda \rangle$ ; hence, there is a canonical such sequence, which belongs to every substructure  $M \prec \langle H(\Omega), \in, \dots \rangle$  to which the parameters  $\kappa$  and  $\lambda$  belong.

Denote  $\lambda = \max \text{pcf}(\{\aleph_n : n \in \omega\}) = \text{cov}(\aleph_\omega, \aleph_0)$  and recall that  $\lambda$  is regular. Let  $U$  be the canonical filter such that  $\text{tcf}(\prod_{n < \omega} \aleph_n / U) = \text{cov}(\aleph_\omega, \aleph_0)$  and let  $\langle f_\alpha : \alpha < \lambda \rangle \in (\prod_n \aleph_n)^\lambda$  be the canonical  $\lambda$ -scale for  $<_U$ . Fix a continuously increasing chain  $\bar{M} := \langle M_i : i < \lambda \rangle$  of elementary submodels of  $\langle H(\Omega), \in, \dots \rangle$  satisfying the following for all  $i < \lambda$ .

- $|M_i| < \lambda$ ,
- $i + 1 \subseteq M_i$ ,



- $\overline{M} \upharpoonright (i+1) \in M_{i+1}$ .

Let  $E \subseteq \lambda$  be the club set of points  $i < \lambda$  for which  $M_i \cap \lambda = i$ . The sequence  $\langle f_i : i \in E \rangle$  is a  $\lambda$ -scale. Finally, set  $\mathcal{F} = \{\text{ran } f_i : i \in E\}$ .

To prove that  $\mathcal{F}$  is  $(\aleph_4, \aleph_1)$ -sparse, let  $A \in [E]^{\aleph_4}$  be given of order-type  $\omega_4$ , and we shall find some  $B \in [A]^{\aleph_1}$  such that  $|\bigcup_{j \in B} \text{ran } f_j| = \aleph_1$ . By Proposition 3.19, it suffices that the  $B$  we find be such that  $\overline{f} \upharpoonright B$  is flat.

Fix a continuously increasing chain  $\overline{N} = \langle N_\zeta : \zeta \leq \omega_3 \rangle$  of elementary submodels of  $\langle H(\Omega), \in, \dots \rangle$  satisfying the following for all  $\zeta \leq \omega_3$ .

- $|N_\zeta| = \aleph_3$ ,
- $\{\overline{M}, A, E\} \cup \omega_3 \subseteq N_0$ ,
- $\overline{N} \upharpoonright (\zeta + 1) \in N_{\zeta+1}$ .

Let  $h(\zeta) = \sup(N_\zeta \cap A)$  for all  $\zeta \leq \omega_3$ . As  $A$  has order-type  $\omega_4$  and  $N_\zeta$  has cardinality  $\omega_3$ ,  $h(\zeta) < \sup A$  for all  $\zeta \leq \omega_3$ . Also, as  $A, E \in N_\zeta$ , it follows that  $h(\zeta) \in E$  and is a limit point of  $A$ , for every  $\zeta \leq \omega_3$ . For  $\zeta \leq \omega_3$  let  $j(\zeta) = \min\{A \setminus h(\zeta)\}$ . So

$$(3.1) \quad h(\zeta) \leq j(\zeta) < h(\zeta + 1)$$

for all  $\zeta < \omega_3$ , by elementarity.

In the model  $M_{h(\omega_3)+1}$ , there exists some canonical function  $g : \omega_3 \rightarrow h(\omega_3)$  which is increasing and continuous and has range cofinal in  $h(\omega_3)$ . Let  $C \subseteq \omega_3$  be the club set of points  $\zeta < \omega_3$  which satisfy  $h(\zeta) = g(\zeta)$ , and let  $\delta \in E_{\omega_1}^{\omega_3}$  be such that  $c_\delta \subseteq C$  where  $\langle c_\eta : \eta \in E_{\omega_1}^{\omega_3} \rangle$  is the canonical club guessing sequence. Let  $B = \{j(\xi) : \xi \in c_\delta\}$ . As  $\text{otp}(c_\delta) = \omega_1$  and  $\zeta \mapsto j(\zeta)$  is order-preserving,  $B \in [A]^{\aleph_1}$ . We prove that  $\overline{f} \upharpoonright B$  is flat. By (3.1), it suffices to prove that  $\langle f_{h(\xi)} : \xi \in c_\delta \rangle$  is flat.

**Claim.**  $\sup_{\rho \geq \xi \in c_\delta} f_{h(\xi)} <_U f_{h(\rho+1)}$  for all  $\rho \in c_\delta$ .

*Proof.* Let  $t = \sup_{\rho \geq \xi \in c_\delta} f_{h(\xi)}$ . As the sequence  $\langle h(\xi) : \xi \in c_\delta \cap (\rho+1) \rangle$  belongs to  $N_{\rho+1}$ , also  $t \in N_{\rho+1}$ . Since  $c_\delta \subseteq C$ , we have  $h(\xi) = g(\xi)$  for all  $\xi \in c_\delta \cap (\rho+1)$ . Since  $g, c_\delta, \rho \in M_{h(\omega_3)+1}$  and  $g(\xi) = h(\xi)$  for all  $\xi \in c_\delta$ , the set  $\{h(\xi) : \xi \in c_\delta \cap (\rho+1)\}$  also belongs to  $M_{h(\omega_3)+1}$ . Therefore,  $t \in M_{h(\omega_3)+1}$ ; hence,  $t <_U f_\gamma$  where  $\gamma = \sup(M_{h(\omega_3)+1} \cap \lambda)$ .

Observe that  $\gamma < \sup(A)$  because  $A \subset E$ . Therefore,  $\alpha = \min(A \setminus \gamma)$  witnesses the truth of the sentence “There exists  $a \in A$  such that  $t <_U f_a$ .” As  $t, A, \overline{f} \in N_{\rho+1}$ , we can find such an  $a$  in  $A \cap N_{\rho+1}$  by elementarity. Consequently, there exists some  $\beta < h(\rho+1)$  such that  $t <_U f_\beta$ . Hence,  $t <_U f_{h(\rho+1)}$ .  $\square$

$\square$

Menachem Magidor pointed out to us that as every point of cofinality  $\aleph_4$  in the scale which is fixed in the proof of Theorem 3.20 is flat, the family constructed there in has the property that every subfamily of size  $\aleph_4$  contains a subset of size  $\aleph_4$  which is free. In view of Proposition 3.13, this follows directly from  $(\aleph_4, \aleph_1)$ -sparseness of the family.

**3.2.3. A refinement.** A refinement of Theorem 3.20 can be proved as follows, using the trick of Main Claim 1.3 and Claim 1.4 in chapter 2 of [31]. By stretching the sequence of models  $\overline{N}$  to length  $\aleph_4$ , one gets that every point of cofinality  $\aleph_4$  in  $\overline{f}$  above is flat. Suppose  $\langle f_\alpha : \alpha < \aleph_4 \rangle$  is  $(<_U$ -increasing and) flat and fix  $\langle h_\alpha : \alpha < \omega_4 \rangle$  which is  $<$ -increasing and equivalent to  $\overline{f}$ , that after thinning out

and re-numbering, it holds that  $f_\alpha < h_\alpha <_U f_{\alpha+1}$  for every  $\alpha < \omega_4$ . For each  $\alpha < \omega_4$  let  $A_\alpha \in U$  be such that  $h_\alpha \upharpoonright A_\alpha < f_{\alpha+1} \upharpoonright A_\alpha$ . If  $U$  is generated by fewer than  $\aleph_4$  sets, then for an unbounded set of  $\alpha < \omega_4$  the  $A_\alpha$  can be chosen as a fixed set  $B \in U$ , and then  $f_{\alpha+1} \upharpoonright A_\alpha$  are pairwise disjoint functions. This means that the family of countable sets  $\{\text{ran } f_\alpha : \alpha \in A\}$  contains a subfamily  $\{\text{ran } f_{\alpha+1} : \alpha \in A'\}$  of the same size with a disjoint refinement — for each  $\alpha \in A'$  the set  $\text{ran}(f_{\alpha+1} \upharpoonright B)$  is a countably infinite subset of  $\text{ran } f_{\alpha+1}$  and these sets are pairwise disjoint.

The ideal  $J_{<\max\text{pcf}}$  is generated by fewer than  $\aleph_4$  PCF generators by the pcf theorem. Therefore, setting  $U = J_{<\max\text{pcf}}^*$ , we have the following theorem:

**Theorem 3.22.** *There is a cofinal  $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$  with the property that every  $A \in [\mathcal{F}]^{\aleph_4}$  contains  $A'$  of size  $\aleph_4$  which has a disjoint refinement.*

**3.3. Consistency results.** Our main aim in the rest of the paper is to show, modulo very large cardinals, that the value of  $\text{Nt}((2^{\aleph_\omega})_\delta)$  is undecidable by the usual axioms of set theory. We start by proving the consistency of it being the minimum possible value.

**Lemma 3.23.** *(Shelah, [31]) If  $\text{cov}(\aleph_\omega, \aleph_0) = \aleph_{\omega+1}$  and there exists a good scale of size  $\aleph_{\omega+1}$  in  $(\prod_{n \in \omega} \aleph_n, \leq^*)$ , then there is an  $(\aleph_1, \aleph_1)$ -sparse cofinal family which is cofinal in  $[\aleph_\omega]^\omega$ .*

*Proof.* Let  $\bar{f} = \{f_\alpha : \alpha < \aleph_{\omega+1}\}$  be a good scale in  $(\prod_{n \in \omega} \aleph_n, \leq^*)$ . We claim that  $\{\text{ran } f_\alpha : \alpha < \aleph_{\omega+1}\}$  is an  $(\aleph_1, \aleph_1)$ -sparse family. Indeed, let  $\{f_{\alpha_i} : i < \omega_1\} \subset \bar{f}$ , where  $\{\alpha_i : i < \omega_1\}$  is an increasing sequence of ordinals. Then  $\gamma = \sup_{i < \omega_1} \alpha_i$  has cofinality  $\aleph_1$  and hence the sequence  $\bar{f} \upharpoonright \gamma$  is flat. By Proposition 3.19,  $\bigcup_{i < \omega_1} \text{ran } f_{\alpha_i}$  is uncountable.  $\square$

**Corollary 3.24.** *If  $\square_{\aleph_\omega}$  and  $\text{cov}(\aleph_\omega, \omega) = \aleph_{\omega+1}$  hold then  $\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$ .*

*Proof.* From  $\square_{\aleph_\omega}$  follows that there is a good scale (actually, “very good scale”, see Theorem 4 in [9]) of length  $\aleph_{\omega+1}$  on  $(\prod_{n \in A} \aleph_n, \leq^*)$  for some infinite  $A \subseteq \omega$ . The proof of Lemma 3.23 can be trivially modified to accommodate the restriction of the index set to  $A$ .  $\square$

The statement  $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$  is known as *Chang’s Conjecture for  $\aleph_\omega$* . Assuming the consistency of slightly more than a huge cardinal, Chang’s Conjecture for  $\aleph_\omega$  is consistent with the GCH by [23]. If Chang’s Conjecture for  $\aleph_\omega$  holds, then no family of countable subsets of  $\aleph_\omega$  whose size is  $> \aleph_\omega$  can be  $(\aleph_1, \aleph_1)$ -sparse. Therefore we have the following theorem due to Lajos Soukup.

**Theorem 3.25.** *([35]) Assume Chang’s Conjecture for  $\aleph_\omega$ . Then  $\text{Nt}((2^{\aleph_\omega})_\delta) \geq \aleph_2$ . If CH is also assumed, then  $\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_2$ .*

We do not know whether it is consistent that  $\pi\text{Nt}((2^{\aleph_\omega})_\delta) \geq \aleph_2$ . This seems to require completely different techniques. Indeed, we can prove that  $\pi\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$  is consistent with Chang’s Conjecture for  $\aleph_\omega$ .

**Lemma 3.26.** *If  $\mathbb{P}$  is a ccc partial order then forcing with  $\mathbb{P}$  preserves Chang’s Conjecture at  $\aleph_\omega$  (and everywhere else).*

*Remark.* The above lemma is apparently well-known, and mentioned without proof by Shelah [32]. We include a proof since we failed to find one in the literature.

*Proof.* An equivalent formulation of Chang's Conjecture at  $\aleph_\omega$  is that for all sufficiently large regular  $\theta$  and all  $A \in H(\theta)$ , there exists  $M \prec H(\theta)$  such that  $A \in M$ ,  $|M \cap \aleph_{\omega+1}| = \aleph_1$ , and  $|M \cap \aleph_\omega| = \aleph_0$ .

Assume Chang's Conjecture at  $\aleph_\omega$  and let  $G$  be a  $V$ -generic filter of  $\mathbb{P}$ . Choose  $\theta$  large enough that  $\mathbb{P}, \aleph_{\omega+1} \in H(\theta)$ . In  $V[G]$ , let  $A \in H(\theta)$ . Back in  $V$ , let  $\dot{A}$  be a  $\mathbb{P}$ -name for  $A$  and let  $N \prec H(\theta)$  be such that  $\dot{A}, \mathbb{P} \in N$ ,  $|N \cap \aleph_{\omega+1}| = \aleph_1$ , and  $|N \cap \aleph_\omega| = \aleph_0$ . We claim that  $G$  is also  $N$ -generic. Indeed, let  $C \in N$  be a maximal antichain. Since  $|C| \leq \aleph_0$ , we have that  $C \subset N$ . Now, since  $G$  is  $V$ -generic, there is  $x \in G \cap C \subset N$ , which proves the claim. Set  $M = N[G](= \{\tau_G : \tau \in V^{\mathbb{P}} \cap N\})$ . Since  $\mathbb{P} \in N$ , we have  $M \prec H(\theta)[G]$ . We have  $M \cap \theta = N \cap \theta$  because  $\mathbb{P}$  is  $N$ -generic. Hence, in  $V[G]$  we have  $M \prec H(\theta)$ ,  $A \in M$ ,  $|M \cap \aleph_{\omega+1}| = \aleph_1$ , and  $|M \cap \aleph_\omega| = \aleph_0$ , as desired. (Note that the above argument generalizes to any Chang conjecture  $(\kappa, \lambda) \rightarrow (\mu, < \nu)$ .)  $\square$

**Theorem 3.27.** *There is a model of Chang's Conjecture for  $\aleph_\omega$  where  $\pi\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$ .*

*Proof.* Assume GCH plus Chang's Conjecture (at  $\aleph_\omega$ ) in the ground model and force with finite partial functions on  $\aleph_{\omega+1}$ . Then, in a generic extension,  $\mathfrak{c} = \aleph_{\omega+1} = 2^{\aleph_\omega}$  and Chang's Conjecture still holds by Lemma 3.26. Moreover,  $(2^{\aleph_\omega})_\delta$  is homeomorphic to  $((2^\omega)_\delta)^{\aleph_\omega}$ , which in turn is homeomorphic to  $(D(\aleph_{\omega+1}))^{\aleph_\omega}$ , where  $D(\aleph_{\omega+1})$  denotes the discrete space of size  $\aleph_{\omega+1}$ . We now prove that in a generic extension  $\pi\text{Nt}((D(\aleph_{\omega+1}))^{\aleph_\omega}) = \aleph_1$ . Indeed, let  $\{\sigma_\alpha : \alpha < \aleph_{\omega+1}\}$  be a cofinal family of countable partial functions from  $\aleph_\omega$  to  $\aleph_{\omega+1}$ . For every  $\alpha < \aleph_{\omega+1}$ , choose  $\beta_\alpha \notin \text{dom}(\sigma_\alpha)$ . Define  $\mathcal{F} = \{\sigma_\alpha \cup \langle \beta_\alpha, \alpha \rangle : \alpha < \aleph_{\omega+1}\}$ , which is a cofinal family. Suppose by contradiction that  $\langle \mathcal{F}, \supseteq \rangle$  is not  $\omega_1^{op}$ -like. Then there is an uncountable set  $A \subset \aleph_{\omega+1}$  and a countable partial function  $\tau$  such that  $\sigma_\alpha \cup \langle \beta_\alpha, \alpha \rangle \subset \tau$  for every  $\alpha \in A$ . If the  $\beta_\alpha$ s are all distinct then  $\tau$  has uncountable domain, while if there are distinct  $\alpha, \gamma \in A$  such that  $\beta_\alpha = \beta_\gamma$ , then  $\tau$  is not a partial function.  $\square$

*Remark.* The proof above shows that  $2^\omega = (\aleph_\omega)^{\aleph_0}$  implies  $\pi\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$ .

**Corollary 3.28.** *There is a model of ZFC where  $\pi\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1 < \aleph_2 = \text{Nt}((2^{\aleph_\omega})_\delta)$*

In contrast with the above separation of Noetherian type and Noetherian  $\pi$ -type, the weight and  $\pi$ -weight of  $(2^{\aleph_\omega})_\delta$  are both  $(\aleph_\omega)^{\aleph_0}$  in every model of ZFC.

We would still be interested in examples showing the sharpness of Theorem 3.2 using milder set-theoretic assumptions.

*Question 3.29.* Is the existence of a compact space  $X$  such that  $\text{Nt}(X_\delta) > 2^{\text{Nt}(X)}$  equiconsistent with ZFC?

*Question 3.30.* Is it consistent that  $\text{Nt}((2^{\aleph_\omega})_\delta) > \aleph_2$ ?

Question 3.30 is related to approachability. Given a sequence  $\langle C_i : i < \lambda \rangle$  where  $C_i \subseteq i$  is unbounded, and, for club many  $i$ ,  $\text{otp}(C_i) = \text{cf}(i)$ , an ordinal  $i < \aleph_{\omega+1}$  is approachable with respect to  $\bar{C}$  if  $\{C_i \cap j : j < i\} \subseteq \{C_j : j < i\}$ . As argued by Foreman and Magidor in the proof of Claim 4.4 of [12], for every  $\bar{C}$  as above and every continuous scale  $\langle f_i \rangle_{i < \lambda}$  of a reduced product  $\prod_{n < \omega} \aleph_n / U$ , there is a club  $D \subseteq \lambda$  such that if  $i \in D$  is approachable with respect to  $\bar{C}$ , then  $\bar{f}$  is flat at  $i$ .

Therefore, if we could find a club  $E$  and  $\overline{C}$  to which every  $\alpha \in E \cap E_{\omega_2}^{\text{cov}(\aleph_\omega, \omega)}$  is approachable, then we could deduce  $\text{Nt}((2^{\aleph_\omega})_\delta) \leq \aleph_2$ , arguing as in the proof of Lemma 3.23. Foreman and Magidor asked a related question, whether ZFC+GCH implies a version of Very Weak Square for  $E_{\omega_2}^{\aleph_{\omega+1}}$ .

Sharon and Viale [30] have shown that MM implies that club many points in  $E_{>\omega_1}^{\aleph_{\omega+1}}$  are approachable. Now, MM implies that  $\text{cov}(\aleph_\omega, \aleph_0) = (\aleph_\omega)^{\aleph_0} = \aleph_{\omega+1}$ . (See [13], Theorem 10 and Corollary 11.) Thus, MM implies  $\text{Nt}((2^{\aleph_\omega})_\delta) \leq \aleph_2$ .

*Question 3.31.* Does MM imply that  $\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_2$ ?

A positive answer would reduce the consistency strength thus far required to break  $\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$ , for the consistency of Martin's Maximum has been proved relative to a supercompact cardinal [13]. Mild evidence for a positive answer is provided by Magidor's result that MM negates the existence of good scales (see [8], Theorem 17.1, for a proof).

We also remark that the consistency of Chang conjecture's variant  $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_2, \aleph_1)$  would imply the consistency of  $\text{Nt}((2^{\aleph_\omega})_\delta) > \aleph_2$ . Indeed, it implies that every cofinal family of countable subsets of  $\aleph_\omega$  contains  $\aleph_2$ -many members whose union  $U$  has size  $\aleph_1$ . But then, by the pigeonhole principle,  $\aleph_2$ -many of them would have to be contained in an initial segment of  $U$  (according to some ordering of type  $\omega_1$ ). Thus, by Theorem 3.14, we would have  $\text{Nt}((2^{\aleph_\omega})_\delta) > \aleph_2$ . However, the consistency of this version of Chang's Conjecture alone is an open problem [30].

Although  $\text{Nt}((2^{\aleph_\omega})_\delta) = \aleph_1$  is not consistent with Chang's Conjecture for  $\aleph_\omega$ , it is certainly consistent with very large cardinals. Let us note three reasons for that. First, it is standard that we can add a  $\square_{\aleph_\omega}$ -sequence (and force GCH at  $\aleph_\omega$ ) with a mild forcing (*i.e.*, a forcing smaller than any large cardinal). Second, we can directly produce an  $(\aleph_1, \aleph_1)$ -sparse cofinal family with a mild forcing. Assume that  $\mathfrak{c} < \aleph_\omega$  in the ground model (or force it) and let  $\mathbb{P} = [[\aleph_\omega]^\omega]^\omega$  with  $q \leq p$  iff  $q \supseteq p$  and  $y \not\subseteq x$  for all  $x \in p$  and  $y \in q \setminus p$ . If  $G$  is a  $V$ -generic filter of  $\mathbb{P}$ , then  $\mathcal{F} = \bigcup G$  is cofinal in  $([\aleph_\omega]^\omega)^V$ . Since  $\mathbb{P}$  is countably closed and has the  $\mathfrak{c}^+$ -cc,  $([\aleph_\omega]^\omega)^{V[G]} = ([\aleph_\omega]^\omega)^V$ , so  $\mathcal{F}$  is actually cofinal in the  $[\aleph_\omega]^\omega$  of  $V[G]$ . Therefore, for every  $x \in [\aleph_\omega]^\omega$  we can find  $y, p$  with  $x \subseteq y \in p \in G$ , which implies  $\mathcal{F} \cap \mathcal{P}(x) \subseteq p$ . Thus,  $\mathcal{F}$  is  $(\aleph_1, \aleph_1)$ -sparse.

Third, the combinatorial principle *Very Weak Square* of Foreman and Magidor [12] implies that a continuous scale contains a club set of functions such that every function indexed by an ordinal of cofinality  $\omega_1$  is a flat point ([12], Claim 4.4). So if we restrict ourselves to that club set of points, using the same argument of Lemma 3.23, we get an  $(\aleph_1, \aleph_1)$ -sparse cofinal family of countable subsets of  $\aleph_\omega$ . Now, by Theorem 2.5 of [12], if  $\kappa$  is supercompact in a model  $M$  of GCH, there is a generic extension of  $M$  in which cardinals and cofinalities are preserved, Very Weak Square holds at the successor of every singular cardinal, and  $\kappa$  remains supercompact. Thus, Corollary 3.24 can be generalized to show that  $\forall \alpha \text{ Nt}((2^{\aleph_{\alpha+\omega}})_\delta) = \aleph_1$  is consistent with the existence of a supercompact cardinal.

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