BETWEEN HOMEOMORPHISM TYPE AND TUKEY TYPE

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Abstract. Call a compact space X *pin homogeneous* if every two points a, b are *pin equivalent*, meaning that there exists a compact space Y, a quotient map $f\colon Y \to X$, and a homeomorphism $g\colon Y \to Y$ such that $gf^{-1}\{a\} = f^{-1}\{b\}$. We will prove a representation theorem for pin equivalence; transitivity of pin equivalence will be a corollary.

Pin homogeneity is strictly weaker than homogeneity and pin equivalence is strictly stronger than Tukey equivalence. Just as with topological homogeneity, no infinite compact F-space is pin homogeneous. On the other hand, $X \times 2^{\chi(X)}$ is pin homogeneous for every compact X. And there is a compact pin homogeneous space with points of different π-character.

1. Introduction

In this paper, all spaces are assumed to be Hausdorff.

Products, but apparently not much else, preserve both compactness and homogeneity[1] of topological spaces. Meanwhile, compact topological groups are ccc and Čech-Stone remainders of infinite discrete spaces are not homogeneous. For essentially these reasons, "large" homogeneous compact spaces are hard to come by. Van Douwen's Problem, asked no later than 1980 and still open in all models of ZFC [1], asks whether there is a homogeneous compact space with $\mathfrak{c}^+$-many disjoint open sets. This a special case of a very natural question:

Question 1.1. Is every compact space X a continuous image of some homogeneous compact space Y? [7]

Even for some important spaces X without $\mathfrak{c}^+$-many disjoint open sets, including $\omega_1 + 1$, $\beta\omega$, and $\beta\omega \setminus \omega$, the above question is open as far as I know. Two significant partial results are:

(1) (Motorov [11]) If X is first countable, compact,[2] and zero-dimensional, then X^ω is homogeneous.
(2) (Kunen [6]) No product of one or more infinite compact F-spaces and zero or more spaces with character less than $\mathfrak{c}$ is homogeneous.

Approaching Question 1.1 less directly, we can consider weaker forms of homogeneity. For example, say that a space is *Tukey homogeneous* if every two points have Tukey equivalent neighborhood filters. This is a much weaker than homogeneity. For example, the compact space $2^\omega \times 2^{\omega_1}_{\mathrm{lex}}$ is Tukey homogeneous yet has

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[1] A space X is *homogeneous* if for each pair $(a, b) \in X^2$ some autohomeomorphism $f\colon X \to X$ sends a to b.

[2] Dow and Pearl [3] showed that compactness is not needed here.

points with different π-characters. And, in striking contrast to Kunen's theorem, $X \times 2^{\chi(X)}$ is Tukey homogeneous for every infinite space X. [8] And though $\beta\omega \setminus \omega$ is Tukey inhomogeneous under the assumption of $\mathfrak{d} = \mathfrak{c}$ (and more generally in any model of set theory where $\beta\omega \setminus \omega$ has a P-point), whether ZFC alone proves this inhomogeneity is a significant open problem in its own right, equivalent [9] to Isbell's Problem:

Question 1.2. Is it consistent with ZFC that $(\mathcal{U}, \supset)$ is Tukey equivalent to $([\mathfrak{c}]^{<\aleph_0}, \subset)$ for every free ultrafilter $\mathcal{U}$ on ω? [4, 2]

This paper introduces pin equivalence, a strict strengthening of Tukey equivalence of points in compact spaces. Pin equivalence enjoys a representation theorem in terms of closed binary relations and, through Stone duality, an appealing Boolean algebraic interpretation. We will also show that, as in the case of topological homogeneity, no infinite compact F-space is pin homogeneous. On the other hand, as in the Tukey case, $X \times 2^{\chi(X)}$ is pin homogeneous for all compact X, as is every first countable crowded compact X. We will also show that $2^\omega \times 2^{\omega_1}_{\mathrm{lex}}$ is pin homogeneous despite having points with different π-characters. Thus, $2^\omega \times 2^{\omega_1}_{\mathrm{lex}}$ is a pin homogeneous space with all points of Tukey type $\omega \times \omega_1$, which I count as a tiny bit of progress towards answering a question I have asked before:

Question 1.3. Is there a compact homogeneous space with points of Tukey type $\omega \times \omega_1$? [10]

In every known example of a compact homogeneous space X, the (neighborhood filters of) points are Tukey equivalent to $([\chi(X)]^{<\aleph_0}, \subset)$. [8]

Without further ado, pin equivalence defined:

Definition 1.4.

- Call closed sets A, B in a compact space X *pin equivalent* and write $A \equiv_{\mathrm{p}} B$ if there exist a compact space Y, a continuous surjection $f\colon Y \to X$,[3] and a homeomorphism $g\colon Y \to Y$ such that $gf^{-1}A = f^{-1}B$.
- Call points a, b in a compact space X pin equivalent and write $a \equiv_{\mathrm{p}} b$ if $\{a\} \equiv_{\mathrm{p}} \{b\}$.
- Call a compact space *pin homogeneous* if all pairs of points are pin equivalent.

(Pin equivalence is transitive, but not obviously so. Wait for the proof.)

Observe that, without loss of generality, f may be assumed invertible at a and b because we may replace Y with its quotient where $f^{-1}\{a\}$ and $f^{-1}\{b\}$ are collapsed to points. This leads to my motivation for "pin." I visualize X inflated to a continuous preimage Y, but with a and b pinned down.

Example 1.5. Closed intervals are pin homogeneous. To see why, let us show that $0 \equiv_{\mathrm{p}} 2$ in $X = [-2, 2]$. Let A be the hollow diamond

$$\{(x, y) \in [-2, 2]^2 \mid |x| + |y| = 2\}.$$

Truncate A to $B = A \cap [-1, 2]^2$ and then extend to $Y = B \cup [-2, -1]^2$. Then $f(x, y) = x$ defines a continuous surjection from Y to X and $g(x, y) = (y, x)$ defines a continuous involution of Y such that

$$gf^{-1}\{0\} = g\{(0, 2)\} = \{(2, 0)\} = f^{-1}\{2\}.$$

[3] Between compact Hausdorff spaces, all continuous surjections are quotient maps.

We will show later that every instance of pin equivalence in an arbitrary compact X is also witnessed by a symmetric subspace of X^2.

Definition 1.6. A space is *Boolean* if is compact and has a base consisting of clopen sets.

Restricting the definition of pin equivalence to Boolean spaces and applying Stone duality, we obtain and algebraic version of pin equivalence that is very natural: two filters are pin equivalent if they generate isomorphic filters in some larger Boolean algebra. More precisely:

Definition 1.7. Given two filters F, G of a Boolean algebra A, we say that F and G are *pin equivalent* in A and write $F \equiv_{\mathrm{p}} G$ if there is a Boolean algebra B extending A and there is a (Boolean) automorphism h of B that sends the filter of B generated by F to the filter of B generated by G.

Next observe that if f in Definition 1.4 is required to be a homeomorphism instead of a mere continuous surjection, then pin homogeneity becomes homogeneity. This suggests a strategy for incremental progress towards solving the open problem of whether every compact space is a quotient of a homogeneous compact space: start with the positive solution to the analogous problem for pin homogeneous compacts and incrementally require more of f.

Question 1.8. How much can we strengthen pin homogeneity before the analog of Question 1.1 for this intermediate homogeneity concept becomes as hard as Question 1.1 itself?

A natural strengthening of "continuous surjection" is "open continuous surjection." So, let us define *open pin equivalence* and *open pin homogeneity* by the requirement that f to also be an open map. Open pin homogeneous compact spaces appear more difficult to obtain. In particular, open pin equivalence is easily seen to preserve π-character.

Question 1.9. Is every compact space a continuous image of an open pin homogeneous compact space?

2. A representation theorem

To my mind, the best evidence so far that pin equivalence is worth studying is the following representation theorem.

Theorem 2.1. *Points a, b in compact space X are pin equivalent iff there is a symmetric binary relation R with domain X such that R is closed in X^2 and, for all $x \in X$, we have $aRx \Leftrightarrow x = b$ and $bRx \Leftrightarrow x = a$.*

Corollary 2.2. *If a, b are pin equivalent in a compact space X, then this is witnessed by f, g, Y where Y is a closed symmetric subset of X^2, $f\colon Y \to X$ is the first coordinate projection, and $g\colon Y \to Y$ is the continuous involution $(x, y) \mapsto (y, x)$.*

Before proving Theorem 2.1, we establish a version of the tube lemma for fibers.

Lemma 2.3. *Suppose Y is compact, $f\colon Y \to X$ is continuous, $x \in X$, and V is a neighborhood of $f^{-1}\{x\}$. Then is a neighborhood U of x such that $f^{-1}U \subset V$.*

Proof. Let C be the complement of the interior of V, which is compact and disjoint from $f^{-1}\{x\}$. Then fC is compact and disjoint from $\{x\}$. Let U be the complement of fC, which is a neighborhood of x. Then $f^{-1}U$ is disjoint from C and, therefore, a subset of the interior of V. □

Definition 2.4. Given a binary relation R:

- Let R^{-1} denote the converse relation.
- Given also a set A, let RA denote the set of all b such that aRb for some $a \in A$.
- Given also a binary relation S, let SR denote the set of all pairs (a, c) such that $aRbSc$ for some b.

Proof of Theorem 2.1. Suppose $a \neq b$, Y is compact, $f\colon Y \to X$ is a quotient map, $g\colon Y \to Y$ is a homeomorphism, $\{c\} = f^{-1}\{a\}$, $\{d\} = f^{-1}\{b\}$, and $g(c) = d$. Let us construct R.

Let A_1, B_1 be disjoint neighborhoods of a, b. Choose a neighborhood C_2 of c such that $C_2 \subset f^{-1}A_1$ and $g(C_2) \subset f^{-1}B_1$. Applying Lemma 2.3, choose a closed neighborhood A_3 of a such that $f^{-1}A_3 \subset C_2$. Let $B_3 = fgf^{-1}A_3$, which is compact because Y is compact. Applying Lemma 2.3 again, B_3 is a neighborhood of b. Also, A_3 and B_3 are disjoint because

$$A_3 \subset fC_2 \subset A_1 \text{ and } B_3 = fgf^{-1}A_3 \subset fgC_2 \subset B_1.$$

Let T be the all $(p, q, r) \in X \times X \times Y$ such that $f(r) = p$ and $f(g(r)) = q$. This set is compact. Let S be the set of all $(p, q) \in A_3 \times B_3$ such that $(p, q, r) \in T$ for some $r \in Y$. This set is also compact. And, since $B_3 = fgf^{-1}A_3$, the domain and range of S are A_3 and B_3. Moreover, a is the unique p satisfying pSb and b is the unique q satisfying aSq. Finally, let D be the closure of the complement of $A_3 \cup B_3$. Then $R = S \cup S^{-1} \cup D^2$ is as desired. □

Definition 2.5. Given a, b, X, R as in Theorem 2.1, we say that R *represents* $a \equiv_{\mathrm{p}} b$ in X.

Our representation theorem helps us prove several nice properties of pin equivalence, starting with the next lemma, which we will use many times. This lemma allows us to use a relation R as above like a function that is continuous at a and b.

Lemma 2.6. *If R represents $a \equiv_p b$ in compact space X and V is a neighborhood of b, then a has a neighborhood U such that $RU \subset V$.*

Proof. Suppose not. Then there are nets $(p_i)_{i\in I}$, $(q_i)_{i\in I}$ in X such that $p_i \to a$, p_iRq_i, and $q_i \notin V$. Since X is compact, $(q_i)_{i\in I}$ has a cluster point c. And $c \neq b$ since c is not in the interior of V. Since R is closed, aRc. But this contradicts $aRx \Leftrightarrow x = b$. □

Corollary 2.7. *Suppose R represents $a \equiv_p b$ and net $(x_i)_{i\in I}$ converges to a. If x_iRy_i for all $i \in I$, then $(y_i)_{i\in I}$ converges to b.*

Corollary 2.8. *Suppose, in a compact space, that R represents $a \equiv_p b$, $\mathcal{U}$ is an ultrafilter on a set I, and $\lim_{i\to\mathcal{U}} x_i = a$. If x_iRy_i for all $i \in I$, then $\lim_{i\to\mathcal{U}} y_i = b$.*

Theorem 2.9. *Pin equivalence is transitive.*

Proof. Suppose $a \equiv_{\mathrm{p}} b \equiv_{\mathrm{p}} c$ in compact space X. Let us show that $a \equiv_{\mathrm{p}} c$. We may assume a, b, c are distinct. Let R, S represent $a \equiv_{\mathrm{p}} b$ and $b \equiv_{\mathrm{p}} c$. Let A_1, B_1, C_1 be disjoint closed neighborhoods of a, b, c. Let A_2, C_2 be open neighborhoods of a, c such that $A_2 \subset A_1$, $C_2 \subset C_1$, $RA_2 \subset B_1$, and $SC_2 \subset B_1$. Define relations $\hat{R}, \hat{S}, T$ as follows.

$$\hat{R} = R \cap (X \setminus C_2)^2$$
$$\hat{S} = S \cap (X \setminus A_2)^2$$
$$T = \hat{R}\hat{S} \cup \hat{S}\hat{R} \cup B_1^2$$

Then T is symmetric, $aTx \Leftrightarrow x = c$, and $cTx \Leftrightarrow x = a$. By compactness, T is also closed. We just need to show that T has domain X.

Fix $x \in X$. First, suppose $x \notin A_1 \cup B_1$. We then have $x\hat{S}y$ for some y. Moreover, $y \notin C_2$ because $SC_2 \subset B_1$ and ySx. Therefore, $y\hat{R}z$ for some z. Thus, $x \in \mathrm{dom}(\hat{R}\hat{S})$. If instead $x \notin C_1 \cup B_1$, then $x \in \mathrm{dom}(\hat{S}\hat{R})$ by analogous reasoning. In the only remaining case, $x \in B_1$, we have $x \in \mathrm{dom}(B_1^2)$. Thus, $X = \mathrm{dom}(T)$. □

The next theorem says that pin equivalence is a local property.

Theorem 2.10. *Let Y, Z be closed subspaces of a compact space X. Suppose a, b are in the interior of $Y \cap Z$ and $a \equiv_p b$ in Y. Then $a \equiv_p b$ in Z.*

Proof. Let R represent $a \equiv_{\mathrm{p}} b$ in Y. Let A_1, B_1 be closed neighborhoods of a, b in $Y \cap Z$. Let A_2, B_2 be open neighborhoods of a, b such that $A_2 \subset A_1$, $B_2 \subset B_1$, $RA_2 \subset B_1$, and $RB_2 \subset A_1$. Let $S = R \cap (A_1 \times B_1)$, $T = R \cap (B_1 \times A_1)$, and $U = S \cup T \cup C^2$ where $C = Z \setminus (A_2 \cup B_2)$. Then U represents $a \equiv_{\mathrm{p}} b$ in Z. □

The next lemma isolates a recurring technique from the proofs of the above theorems.

Lemma 2.11. *Given distinct a, b in a compact space X, there exists R that represents $a \equiv_p b$ iff there exists a closed binary relation S on X such that* $\mathrm{dom}(S)$ *is a neighborhood of a,* $\mathrm{ran}(S)$ *is a neighborhood of b disjoint from* $\mathrm{dom}(S)$*, $xSb \Leftrightarrow x = a$, and $aSy \Leftrightarrow y = b$.*

Proof. Given R, let $S = R \cap ((U \times RU) \cup (RV \times V))$ for sufficiently small closed neighborhoods U, V of a, b. Given instead S, let

$$R = S \cup S^{-1} \cup \left(\overline{X \setminus (\mathrm{dom}(S) \cup \mathrm{ran}(S))}\right)^2. \qquad \square$$

3. Pin equivalence vs. Tukey equivalence

Here we show that pin equivalence strictly implies Tukey equivalence.

Definition 3.1. A *directed set* is a nonempty set S equipped with a transitive reflexive relation $\leq$ such that for all $x, y \in S$ there exists $z \in S$ such that $x, y \leq z$.

Definition 3.2. Given two directed sets P, Q:

- We say P is *Tukey below* Q and write $P \leq_{\mathrm{T}} Q$ if there exists $f \colon Q \to P$ that is *convergent*, that is, for every $p_0 \in P$ there exists $q_0 \in Q$ such that $f(q) \geq p_0$ for all $q \geq q_0$.
- We say P is *Tukey equivalent* to Q and write $P \equiv_{\mathrm{T}} Q$ if $P \leq_{\mathrm{T}} Q \leq_{\mathrm{T}} P$.
- A subset U of P is *unbounded* if has no upper bound in P.
- A subset C of P is *cofinal* if for every $p \in P$ has an upper bound in C.

- The *cofinality* $\mathrm{cf}(P)$ of P is least of the cardinalities of cofinal subsets of P.
- Given a cardinal κ, we say P is *κ-directed* if every subset of P size less than κ has an upper bound in P.
- Given a cardinal κ, we say P is *κ-OK* if, for each $f\colon \omega \to P$ there exists $g\colon \kappa \to P$ such that for every $n < \omega$, every increasing n-tuple $\xi_1 < \cdots < \xi_n < \kappa$, and every upper bound $b \in P$ of $\{g(\xi_1), \ldots, g(\xi_n)\}$, we have $f(n) \leq b$.

Below are some elementary consequences of the above definitions.

- Composition preserves convergence.
- If C is a cofinal subset of P, then $C \equiv_{\mathrm{T}} P$.
- If $P \leq_{\mathrm{T}} Q$, then $\mathrm{cf}(P) \leq \mathrm{cf}(Q)$.
- If Q is κ-directed and $P \leq_{\mathrm{T}} Q$, then P is κ-directed.
- If P is λ-OK and $\kappa \leq \lambda$, then P is κ-OK.
- If $\mathrm{cf}(P) \leq \kappa$, then $P \leq_{\mathrm{T}} [\kappa]^{<\aleph_0}$ where $[S]^{<\aleph_0}$ denotes the finite subsets of S ordered by inclusion ($\subset$).

Lemma 3.3. *If P is κ-OK but not ω_1-directed, then $[\kappa]^{<\aleph_0} \leq_{\mathrm{T}} P$.*

Proof. Let f map ω to an unbounded subset of P. Let g be as in the definition of κ-OK. Then g maps each infinite subset of κ to an unbounded set. To obtain a convergent map from P to $[\kappa]^{<\aleph_0}$, map each $p \in P$ to the set of all $\xi < \kappa$ satisfying $g(\xi) \leq p$. □

Through neighborhood filters, the order concepts defined above induce the topological concepts defined next.

Definition 3.4. Given points a, b in space X:

- We denote by $\mathcal{N}_X(a)$ the *neighborhood filter* of a, that is, the set of all $N \subset X$ with a in the interior of N. We make $\mathcal{N}_X(a)$ a directed set by ordering it by containment ($\supset$).
- We say a is Tukey below (resp., Tukey equivalent to) b if $\mathcal{N}_X(a) \leq_{\mathrm{T}} \mathcal{N}_X(b)$ (resp., $\mathcal{N}_X(a) \equiv_{\mathrm{T}} \mathcal{N}_X(b)$).
- We denote by $\chi(a, X)$, the *character* of a, which is the cofinality $\mathrm{cf}(\mathcal{N}_X(a))$ of a's neighborhood filter.
- Given a cardinal κ, we say a is κ-OK if its neighborhood filter is.

Theorem 3.5. *If $a \equiv_p b$ in compact space X, then $a \equiv_{\mathrm{T}} b$.*

Proof. Let R represent $a \equiv_{\mathrm{p}} b$. It suffices to show that $b \leq_{\mathrm{T}} a$. Define $r\colon \mathcal{N}_X(a) \to \mathcal{N}_X(b)$ by $r(U) = RU$. By Lemma 2.6, r is convergent. □

Definition 3.6. Given a point a in a space X:

- We say a is a *P-point* if $\mathcal{N}_X(a)$ is ω_1-directed.
- We say a is a *weak P-point* if $X \setminus C \in \mathcal{N}_X(a)$ for every countable $C \subset X \setminus \{a\}$.

Theorem 3.7. *In a compact space X, if $a \equiv_p b$ and a is not a weak P-point, then neither is b.*

Proof. Let $a \in \overline{\{x_n \mid n < \omega\}}$ but $x_n \neq a$ for all $n < \omega$. Letting some R represent $a \equiv_{\mathrm{p}} b$, choose y_n such that $x_n R y_n$, for each $n < \omega$. Then $y_n \neq b$ for all $n < \omega$. For each $N \in \mathcal{N}_X(a)$, choose $x_{\varphi(N)} \in N$, thus defining a net converging to a. Then $y_{\varphi(N)} \to b$ by Corollary 2.7. Hence, $b \in \overline{\{y_n \mid n < \omega\}}$. □

Kunen proved that the Čech-Stone remainder ω^* has weak P-points, a fact that previously was merely known to be consistent with ZFC. His proof consists of an easy result followed by a hard result:

Lemma 3.8 (Kunen [5]). *In a space, if a point is ω_1-OK, then it is also a weak P-point.*

Lemma 3.9 (Kunen [5]). *In the Čech-Stone remainder ω^*, there is a $\mathfrak{c}$-OK point that is not a P-point.*

Definition 3.10. We say a space X is an *F-space* if every two disjoint open F_σ-sets have disjoint closures.

ω^* is the quintessential example of a compact F-space. Indeed, a Stone space of Boolean algebra is an F-space iff the algebra has the *countable separable property*: every two countably generated ideals I, J with $I \cap J = \{0\}$ extend to principal ideals I', J' with $I' \cap J' = \{0\}$. Now ω^* is homeomorphic to the Stone space of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$. It is an easy exercise to show that this algebra has the countable separable property.

Theorem 3.11. *In the Čech-Stone remainder ω^*, there exist a, b such that $a \equiv_{\mathrm{T}} b$ but $a \not\equiv_p b$.*

Proof. Let $X = \omega^*$. We identify each point $e \in X$ with the ultrafilter

$$\{U \subset \omega \mid e \in \overline{U}\}$$

where the closure $\overline{U}$ is computed in the Čech-Stone compactification $\beta\omega = \omega \cup X$. The map $E \mapsto E \setminus \omega$ surjects from the above ultrafilter to the set of the clopen neighborhoods of e. And for $U, V \subset \omega$, we have $\overline{U} \setminus \omega \subset \overline{V} \setminus \omega$ iff $U \subset^* V$ where $\subset^*$ is inclusion modulo finite sets. Therefore, $e \equiv_{\mathrm{T}} \mathcal{N}_X(e)$ provided e is ordered by $\supset^*$.

Let $a \in X$ be $\mathfrak{c}$-OK but not a P-point. Then $[\mathfrak{c}]^{<\aleph_0} \leq_{\mathrm{T}} a$ by Lemma 3.3 and a is a weak P-point by Lemma 3.8. Let $(c_n)_{n<\omega}$ be a discrete sequence in X and let b be the ultralimit $\lim_{n\to a} c_n$. Then $a \not\equiv_{\mathrm{p}} b$ because b is not a weak P-point.

Claim. $a \leq_{\mathrm{T}} b$.

Proof. We will show that $\varphi(V) = \{n < \omega \mid c_n \in \overline{V}\}$ defines a convergent map from $(b, \supset^*)$ to $(a, \supset)$, noting that the identity map from $(a, \supset)$ to $(a, \supset^*)$ is convergent. Since $(c_n)_{n<\omega}$ is discrete, there are disjoint open F_σ sets $(O_n)_{n<\omega}$ such that $c_n \in O_n$ for each n. Suppose $U \in a$. Since X is an F-space, $\bigcup_{n\in U} O_n$ and $\bigcup_{n\notin U} O_n$ have disjoint closures. Since also $b \in \overline{\bigcup_{n\in U} O_n}$, we may choose $V_0 \in b$ such that $\overline{V_0}$ is disjoint from $\{c_n \mid n \notin U\}$. Therefore, $\varphi(V) \subset U$ for all $V \subset^* V_0$. □

Moreover, $b \leq_{\mathrm{T}} [\mathfrak{c}]^{<\aleph_0}$ since b has cardinality $\mathfrak{c}$. Therefore,

$$a \equiv_{\mathrm{T}} b \equiv_{\mathrm{T}} [\mathfrak{c}]^{<\aleph_0}.$$ □

Remark 3.12. In the above proof, the justification of $a \leq_{\mathrm{T}} b$ works for any $a \in \omega^*$. It really shows that if a is strictly below b in the Rudin-Frolík order, then $(a, \supset)$ is Tukey below $(b, \supset^*)$.

4. Pin inhomogeneity in other F-spaces

Besides ω^*, another simply defined example of a compact F-space is the absolute of 2^ω, that is, the Stone space Ξ of the algebra of regular open subsets of 2^ω. This is an F-space because any regular open algebra is complete. On the other hand, Ξ has a countable π-base because 2^ω does. Therefore, Ξ lacks weak P-points. Hence, our construction of pin-inequivalent points in ω^*, which relied on Kunen's construction of a weak P-point in ω^*, cannot generalize to all infinite F-spaces. Nevertheless, we can use a lemma from another paper of Kunen's to show that every infinite F-space has pin-inequivalent points.

Definition 4.1. Given ultrafilters $\mathcal{U}, \mathcal{V}$ on ω, we say $\mathcal{U}$ is *Rudin-Keisler below* $\mathcal{V}$ and write $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$ if there exists $f\colon \omega \to \omega$ such that $\beta f(\mathcal{V}) = \mathcal{U}$, that is, such that $E \in \mathcal{U}$ iff $f^{-1}E \in \mathcal{V}$, for all $E \subset \omega$.

Theorem 4.2 (Kunen [5])**.** *There are Rudin-Keisler incomparable weak P-points in ω^*.*

Lemma 4.3 (Kunen [6])**.** *Suppose $\mathcal{U}, \mathcal{V}$ are Rudin-Keisler incomparable weak P-points in ω^*. Also suppose that, in a compact F-space X, a is the $\mathcal{U}$-limit of a discrete ω-sequence. Then a is not the $\mathcal{V}$-limit of any ω-sequence in $X \setminus \{a\}$.*

Theorem 4.4. *Let X be an infinite compact F-space. Then there exist $a, b \in X$ such that $a \not\equiv_p b$.*

Proof. Let $\mathcal{U}, \mathcal{V} \in \omega^*$ be Rudin-Keisler incomparable weak P-points. Let $(a_n)_{n<\omega}$ be a discrete sequence in X, let $a = \lim_{n\to\mathcal{U}} a_n$, and let $b = \lim_{n\to\mathcal{V}} a_n$. By Lemma 4.3, b is not the $\mathcal{U}$-limit of any ω-sequence in $X \setminus \{b\}$. Seeking a contradiction, suppose that R represents $a \equiv_{\mathrm{p}} b$. For each $n < \omega$, choose b_n such that $a_n R b_n$. Then $b_n \neq b$ for all $n < \omega$. Also, by Corollary 2.8, $\lim_{n\to\mathcal{U}} b_n = b$. Thus, we have a contradiction. □

5. Pin homogeneity

Definition 5.1. Given a point a in a space X, a subset $\mathcal{S}$ of $\mathcal{N}_X(a)$ is a *neighborhood subbase* at a if $\mathcal{N}_X(a)$ is the smallest filter containing $\mathcal{S}$.

Definition 5.2. A set $\mathcal{E}$ of sets is *independent* if, for each pair of finite nonempty $\mathcal{F}, \mathcal{G} \subset \mathcal{E}$, if $\mathcal{F} \cap \mathcal{G} = \varnothing$, then $\bigcap \mathcal{F} \not\subset \bigcup \mathcal{G}$.

Lemma 5.3. *In a compact space X, if there is a bijection from an independent neighborhood subbase at a to an independent neighborhood subbase at b, then $a \equiv_p b$.*

Proof. We may assume $a \neq b$. Let $f_1\colon \mathcal{A}_1 \to \mathcal{B}_1$ biject from an independent neighborhood subbase at a to an independent neighborhood subbase at b. First, we construct modified $f_1, \mathcal{A}_1, \mathcal{B}_1$ for which $\bigcup \mathcal{A}_1$ and $\bigcup \mathcal{B}_1$ are disjoint. Choose finite nonempty $\mathcal{C}_1 \subset \mathcal{A}_1$ and $\mathcal{D}_1 \subset \mathcal{B}_1$ such that $\bigcap \mathcal{C}_1$ and $\bigcap \mathcal{D}_1$ are disjoint. Let $\mathcal{C}_2 = \mathcal{C}_1 \cup f_1^{-1}\mathcal{D}_1$ and $\mathcal{D}_2 = f_1\mathcal{C}_1 \cup \mathcal{D}_1$. Let

$$\mathcal{A}_2 = \{U \cap \bigcap \mathcal{C}_2 \mid U \in \mathcal{A}_1 \setminus \mathcal{C}_2\};$$

$$\mathcal{B}_2 = \{U \cap \bigcap \mathcal{D}_2 \mid U \in \mathcal{B}_1 \setminus \mathcal{D}_2\}.$$

Then $A = \bigcup \mathcal{A}_2$ and $B = \bigcup \mathcal{B}_2$ are disjoint. Moreover, because $\mathcal{A}_1$ and $\mathcal{B}_1$ are each independent, $\mathcal{A}_2$ and $\mathcal{B}_2$ are too and $f_2(U \cap \bigcap \mathcal{C}_2) = f_1(U) \cap \bigcap \mathcal{D}_2$ defines a bijection from $\mathcal{A}_2$ to $\mathcal{B}_2$.

For each $N \in \mathcal{A}_2$, let

$$T_N = (N \times f_2(N)) \cup ((A \setminus N) \times (B \setminus f_2(N))).$$

Because $\mathcal{A}_2$ and $\mathcal{B}_2$ are each independent, if $\mathcal{F} \subset \mathcal{A}_2$ is finite and nonempty, then $S_{\mathcal{F}} = \bigcap_{N \in \mathcal{F}} T_N$ has domain A and range B. By compactness, $\overline{S_{\mathcal{F}}}$ has domain $\overline{A}$ and range $\overline{B}$; so does $S = \bigcap_{\mathcal{F}} \overline{S_{\mathcal{F}}}$. Moreover, if xT_Nb for some x, N, then $x \in N$. Therefore, xSb implies $x = a$. Likewise, aSy implies $y = b$. By Lemma 2.11, $a \equiv_{\mathrm{p}} b$. □

Theorem 5.4. *If X is a compact space, κ is a cardinal, and $\chi(x, X) \leq \kappa$ for all $x \in X$, then $X \times 2^\kappa$ is pin homogeneous.*

Proof. Given $(a, b) \in X \times 2^\kappa$, it suffices to find an independent local subbase at (a, b) of cardinality κ. Let $\{A_\alpha \mid \alpha < \kappa\}$ be a neighborhood subbase at a. For each $\alpha < \kappa$, let

$$U_\alpha = A_\alpha \times \{y \in 2^\kappa \mid y(\alpha) = b(\alpha)\}.$$

Then $\alpha \neq \beta \Rightarrow U_\alpha \neq U_\beta$ and $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$ is a local subbase at (a, b). To see that $\mathcal{U}$ is independent, suppose $\sigma, \tau \in [\kappa]^{<\aleph_0}$ are disjoint. Define $c \in 2^\kappa$ by $c(\alpha) = b(\alpha)$ iff $\alpha \in \sigma$. Then $(a, c) \in U_\alpha$ for all $\alpha \in \sigma$ and $(a, c) \notin U_\alpha$ for all $\alpha \in \tau$. □

I was not able to adapt the above proof to show that X^κ is pin homogeneous.

Question 5.5. Does every compact space have a pin homogeneous power?

Definition 5.6. A space is *crowded* if it has no isolated points.

Theorem 5.7. *Suppose X is a first countable crowded compact space. Then X is pin homogeneous.*

Proof. Let a, b be distinct points in X. Let $\{A_n \mid n < \omega\}$ and $\{B_n \mid n < \omega\}$ be neighborhood bases at a and b such that $A_n \supsetneq A_{n+1}$ and $B_n \supsetneq B_{n+1}$. Let

$$S = \{(a, b)\} \cup \bigcup_{n < \omega} \left(\overline{A_n \setminus A_{n+1}} \times \overline{B_n \setminus B_{n+1}} \right).$$

By Lemma 2.11, $a \equiv_{\mathrm{p}} b$. □

Proposition 5.8. *Pin homogeneity is productive.*

Proof. Suppose that for each i in some set I we have $Y_i \xrightarrow{g_i} Y_i \xrightarrow{f_i} X_i$ witnessing $a(i) \equiv_{\mathrm{p}} b(i)$ in X_i. Then, letting $X = \prod_i X_i$ and $Y = \prod_i Y_i$, we have $Y \xrightarrow{g} Y \xrightarrow{f} X$ witnessing $a \equiv_{\mathrm{p}} b$ in X where $f(y)(i) = f_i(y(i))$ and $g(y)(i) = g_i(y(i))$. □

6. Pin equivalence and Boolean algebras

The proofs of Lemma 5.3 and Theorem 5.7 implicitly used Boolean isomorphisms between Boolean closures of neighborhood bases. Thus, these results are actually special cases of the following theorem.

Definition 6.1. A *neighborhood subbase* of a subset E of a space X is family $\mathcal{S}$ of subsets of X such that smallest filter containing $\mathcal{S}$ is the set of neighborhoods of E.

Definition 6.2. Given a subset E of a Boolean algebra A, $\langle E\rangle$ denotes the Boolean closure of E.

Theorem 6.3. *Given closed disjoint subsets H, K of a compact space X, we have $H \equiv_p K$ if H and K have neighborhood subbases $\mathcal{U}$ and $\mathcal{V}$ such that there is a map $f\colon \mathcal{U} \to \mathcal{V}$ that extends to a Boolean isomorphism $\varphi\colon \langle\mathcal{U}\rangle \to \langle\mathcal{V}\rangle$ of the Boolean closures of $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{P}(X)$.*

Proof. Let $\varphi\colon \langle\mathcal{U}\rangle \to \langle\mathcal{V}\rangle$ be as above. Choose $U \in \langle\mathcal{U}\rangle$ and $V \in \langle\mathcal{V}\rangle$ such that $H \subset U$, $K \subset V$, and $U \cap V = \varnothing$. Letting $A = U \cap \varphi^{-1}(V)$, we obtain $H \subset A$, $K \subset \varphi(A)$, and $A \cap \varphi(A) = \varnothing$. Let $\mathcal{A}$ be the Boolean subalgebra $\langle\mathcal{U}\rangle \cap \mathcal{P}(A)$ of $\mathcal{P}(A)$ (not a Boolean subalgebra of $\mathcal{P}(X)$); let $\mathcal{B}$ be the Boolean subalgebra $\langle\mathcal{V}\rangle \cap \mathcal{P}(B)$ of $\mathcal{P}(B)$ where $B = \varphi(A)$; let ψ be the restriction of φ to $\mathcal{A}$. Then ψ is a Boolean isomorphism to $\mathcal{B}$.

For each finite partition $\mathcal{E} \subset \mathcal{A}$, the relation

$$T_{\mathcal{E}} = \bigcup_{E\in\mathcal{E}} (E \times \psi(E))$$

has domain A and range B. Moreover, if $\mathcal{F}$ refines $\mathcal{E}$, then $T_{\mathcal{F}} \subset T_{\mathcal{E}}$. By compactness, each $\overline{T_{\mathcal{E}}}$ has domain $\overline{A}$ and range $\overline{B}$; so does $S = \bigcap_{\mathcal{E}} \overline{T_{\mathcal{E}}}$. For any x, if $xT_{\mathcal{E}}y$ for some $y \in K$, then $x \in E$ for the unique $E \in \mathcal{E}$ with $H \subset E$. Therefore, $xSy \in K \Rightarrow x \in H$. Analogously, $H \ni xSy \Rightarrow y \in K$. Let

$$R = S \cup S^{-1} \cup \left(\overline{X \setminus (A \cup B)}\right)^2.$$

Then the involution $g\colon R \to R$ given by $g(x, y) = (y, x)$ and the coordinate projection $f\colon R \to X$ given by $f(x, y) = x$ witness that $H \equiv_{\mathrm{p}} K$. □

Definition 6.4.

- Two filters F, G of a Boolean algebra are *incompatible* if $x \wedge y = 0$ for some $(x, y) \in F \times G$.
- A subset E of a filter F of a Boolean algebra A *generates* F in A is F is the smallest filter of A that contains E.

Corollary 6.5. *Suppose that F and G are incompatible filters of a Boolean algebra and that they are generated by sets D and E. If there is a map from D to E that extends to a Boolean isomorphism from $\langle D\rangle$ to $\langle E\rangle$, then $F \equiv_p G$.*

When compared to Definition 1.7, the converse of Corollary 6.5 looks too good to be true. But I have not yet found a counterexample.

Problem 6.6. Find a Boolean algebra with pin equivalent and incompatible filters F, G such that for all bijections $\varphi\colon D \to E$, if D generates F and E generates G, then φ does not extend to a Boolean isomorphism from $\langle D\rangle$ to $\langle E\rangle$.

We next use the above theorem to show that pin equivalence does not preserve π-character.

Definition 6.7. The *π-character* $\pi\chi(a, X)$ of a point a in a space X is the least of the cardinalities of families $\mathcal{F}$ of nonempty open subsets of X such that every neighborhood of a contains an element of $\mathcal{F}$. Such a family is called a *local π-base* at a.

Definition 6.8. Given an ordinal α, 2^α_{lex} is the set of all $f\colon \alpha \to 2$ with the lexicographic ordering and the associated order topology (which is compact).

In $L = 2^{\omega_1}_{\mathrm{lex}}$, every monotone ω_2-sequence is eventually constant. But every point is the limit of a strictly increasing ω_1-sequence or the limit of a strictly decreasing ω_1-sequence (or both). Moreover, topologically, there are exactly three types of points in L, as shown in the illustration below.

$$\mathrm{II} \xleftarrow[\omega_1]{} \quad \xrightarrow[\omega_1]{} \mathrm{I} \xleftarrow[\omega_1]{} \quad \xrightarrow[\omega]{} \mathrm{III} \xleftarrow[\omega_1]{} \quad \xrightarrow[\omega_1]{} \mathrm{III} \xleftarrow[\omega]{} \quad \xrightarrow[\omega_1]{} \mathrm{II}$$

$2^{\aleph_1}$-many points of L are simultaneously the limit of a strictly increasing ω_1-sequence and the limit of a strictly decreasing ω_1-sequence. Call these points *type I.* Call the two endpoints and the $2^{\aleph_0}$-many points of L with either an immediate predecessor or immediate successor *type II.*[4] All points of type I or II are P-points with Tukey type ω_1 and π-character ω_1. The remainder of L, the set of *type III* points, consists of $2^{\aleph_0}$-many limits of strictly increasing or strictly decreasing ω-sequences. These have Tukey type $\omega \times \omega_1$ and have π-character ω because the nonempty open intervals with endpoints from the ω-sequence form a local π-base.

In the product space $K = 2^\omega \times L$, there are no P-points. But K inherits both π-characters of L; indeed, $\pi\chi((p, q), K) = \pi\chi(q, L)$. On the other hand, every point in K has Tukey type $\omega \times \omega_1$. Interestingly, K is also pin homogeneous.

Definition 6.9. Subalgebras $A_0, \dots A_{n-1}$ of a Boolean algebra B are *independent* if, for all $x \in \prod_{i<n} A_i$, if $x(i) \neq 0$ for all i, then $\bigwedge_{i<n} x(i) \neq 0$.

Theorem 6.10. *$2^\omega \times 2^{\omega_1}_{\mathrm{lex}}$ is pin homogeneous.*

Proof. Continuing to use the above notation K, L, suppose that $x^0 = (p^0, q^0)$, and $x^1 = (p^1, q^1)$ are distinct points in K. For each $i < 2$, conditionally define sets P^i_n, Q^i_α, R^i_n, S^i_α, T^i_n as follows. Let $\{P^i_n \mid n < \omega\}$ be a neighborhood base at p^i such that $P^i_n \supsetneq P^i_{n+1}$. If q^i is type I or II, then let $(Q^i_\alpha)_{\alpha<\omega_1}$ be a sequence of intervals such that $Q^i_\alpha \supsetneq Q^i_\beta$ for $\alpha < \beta$, q^i is in the interior of each Q^i_α, and $\{q^i\} = \bigcap_\alpha Q^i_\alpha$. If q^i is type III, let $(Q^i_\alpha)_{\alpha<\omega_1}$ be a sequence of rays of L such that $Q^i_\alpha \supsetneq Q^i_\beta$ for $\alpha < \beta$ and q^i is the interior of each Q^i_α but on the boundary of $\bigcap_\alpha Q^i_\alpha$. In all cases, let $S^i_\alpha = 2^\omega \times Q^i_\alpha$. If q^i is type I or II, let $T^i_n = P^i_n \times K$. If q^i is type III, let $T^i_n = P^i_n \times R^i_n$ where $(R^i_n)_{n<\omega}$ is a sequence of rays of L such that $R^i_n \supsetneq R^i_{n+1}$ and q^i is the interior of each R^i_n but on the boundary of $\bigcap_n R^i_n$. In all cases,

$$\mathcal{B}^i = \{S^i_\alpha \mid \alpha < \omega_1\} \cup \{T^i_n \mid n < \omega\}$$

is a neighborhood subbase at x^i.

Let $\mathcal{S}^i$ be the Boolean closure of the set of all sets of the form S^i_α. Let $\mathcal{T}^i$ be the Boolean closure of the set of all sets of the form T^i_n. Let $\mathcal{U}^i = \langle \mathcal{S}^i \cup \mathcal{T}^i \rangle$. Since $S^i_\alpha \supsetneq S^i_\beta$ for $\alpha < \beta$, the map $S^0_\alpha \mapsto S^1_\alpha$ extends uniquely to an isomorphism of $\sigma\colon \mathcal{S}^0 \to \mathcal{S}^1$. Likewise, the map $T^0_n \mapsto T^1_n$ extends uniquely to an isomorphism of

[4]Types I and II are topologically distinguishable: each type I point a is in the closure of each of two disjoint topological copies of ω_1 in $L \setminus \{a\}$; the type II points lack this property.

$\tau\colon \mathcal{T}^0 \to \mathcal{T}^1$. Moreover, for each $i < 2$, $\mathcal{S}^i$ and $\mathcal{T}^i$ are independent because, for each $\alpha < \omega_1$ and $n < \omega$, the intersection of $S^i_\alpha \setminus S^i_{\alpha+1}$ and $T^i_n \setminus T^i_{n+1}$ is nonempty because in all cases it contains

$$(P^i_n \setminus P^i_{n+1}) \times (Q^i_\alpha \setminus Q^i_{\alpha+1}).$$

Therefore, $\sigma \cup \tau$ extends uniquely to an isomorphism from $\mathcal{U}^0$ onto $\mathcal{U}^1$. Therefore, by Theorem 6.3, $x^0 \equiv_{\mathrm{p}} x^1$. □

It is not too hard to generalize the above theorem to $\prod_{m\leq n} X_m$ where $X_m = 2^{\omega_m}_{\mathrm{lex}}$ and $n < \omega$. The points of this product space attain all π-characters in $[\omega, \omega_n]$ and have Tukey type $\prod_{m\leq n} \omega_m$. (Note that the product and lexicographic order topologies on 2^ω are identical.)

Theorem 6.11. *For each $n < \omega$, $\prod_{m\leq n} 2^{\omega_m}_{\mathrm{lex}}$ is pin homogeneous.*

Proof. For convenience, let $\omega_{-1} = 1$. Using the above X_m notation, for each $m \leq n$ and $x \in X_m$ there is a least $s(x) \in \{-1, 0, \ldots, m\}$ for which there are two strictly decreasing sequences of rays $(P_\alpha(x) \mid \alpha < \omega_m)$ and $(Q_\beta(x) \mid \alpha < \omega_{s(x)})$ such that each of these rays has x in its interior and

$$\{x\} = \bigcap_{\alpha<\omega_m} P_\alpha(x) \cap \bigcap_{\alpha<\omega_{s(x)}} Q_\alpha(x).$$

Given $y \in Y = \prod_{m\leq n} X_m$, it suffices to show that y has a neighborhood subbase consisting of the union of $n+1$ strictly decreasing chains $(R^m_\alpha \mid \alpha < \omega_m)$ for $m \leq n$ whose respective Boolean closures $\mathcal{A}_0, \ldots, \mathcal{A}_n$ are independent. Letting $y_i = y(i)$, $s_i = s(y_i)$, $P^i_\alpha = P_\alpha(y_i)$, and $Q^i_\alpha = Q_\alpha(y_i)$ for each $i \leq n$, define $R^m_\alpha = \prod_{i\leq n} S^{m,i}_\alpha$ where

$$S^{m,i}_\alpha = \begin{cases} X_i & : i < m \\ P^m_\alpha \cap Q^m_0 & : i = m; s_m = -1 \\ P^m_\alpha & : i = m; 0 \leq s_m < m \\ P^m_\alpha \cap Q^m_\alpha & : i = m; s_m = m \\ X_i & : i > m; s_i \neq m \\ Q^i_\alpha & : i > m; s_i = m, \end{cases}$$

thus making $(R^m_\alpha \mid \alpha < \omega_m)$ strictly decreasing for each $m \leq n$ and the union of these $n+1$ chains a neighborhood subbase at y. Moreover, $\mathcal{A}_0, \ldots, \mathcal{A}_n$ are independent because if $\alpha(m) < \omega_m$ for each $m \leq n$, then $\bigcap_{m\leq n}(R^m_{\alpha(m)} \setminus R^m_{\alpha(m)+1})$ is nonempty because it contains $\prod_{m\leq n}(P^m_{\alpha(m)} \setminus P^m_{\alpha(m)+1})$. (In verifying this, a key observation is that $P^m_{\alpha(m)} \setminus P^m_{\alpha(m)+1} = P^m_{\alpha(m)} \cap Q^m_\beta \setminus P^m_{\alpha(m)+1}$ for all β.) □

On the other hand, it is shown in [10] that if X is a compact space and P and Q are directed sets such that $\mathrm{cf}(P), \mathrm{cf}(Q) \geq \omega$ and Q is $\mathrm{cf}(P)^{++}$-directed, then X has a point not of Tukey type $P \times Q$. In particular, we cannot have a compact space, pin homogeneous or otherwise, with all points of Tukey type $\omega \times \omega_2$.

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References

[1] A. Arhangel′skiĭ, J. van Mill. *Topological Homogeneity.* Recent Progress in General Topology III (K. Hart, J. van Mill, and P. Simon, eds.), Atlantis Press, 2014.

[2] N. Dobrinen. *Survey on the Tukey theory of ultrafilters.* Selected Topics in Combinatorial Analysis, Zbornik Radova, Mathematical Institutes of the Serbian Academy of Sciences, **17(25)** (2015), 53–80.

[3] A. Dow, E. Pearl. *Homogeneity in powers of zero-dimensional first-countable spaces.* Proc. Amer. Math. Soc. **125** (1997), 2503–2510.

[4] J. Isbell. *The category of cofinal types II.* Trans. Amer. Math. Soc. **116** (1965), 394–416.

[5] K. Kunen. *Weak P-points in* $\mathbb{N}^*$, Colloq. Math. Soc. János Bolyai **23** (1978), 741–749. (Reprinted in: *The Mathematical legacy of Eduard Čech* (M. Katětov and P. Simon, eds.), Birkhäuser Verlag (1993), 100–108.)

[6] K. Kunen, *Large homogeneous compact spaces*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland Publishing Co., Amsterdam, 1990, pp. 261–270.

[7] J. van Mill. *Homogeneous compacta.* Open Problems in Topology II (E. Pearl, ed.), Elsevier, 2007, 189–195.

[8] D. Milovich. *Noetherian types of homogeneous compacta and dyadic compacta.* Topology Appl. **156** (2008), 443–464.

[9] D. Milovich, *Tukey classes of ultrafilters on* ω. Topology Proceedings **32** (2008), 351–362.

[10] D. Milovich. *Forbidden rectangles in compacta.* Topology and its Applications **159** (2012), 3180–3189.

[11] D. B. Motorov. *On retracts of homogeneous bicompacta.* Vestnik MGU **5** (1985).

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