What 2^{ω_2} taught me about 2^2

David Milovich
http://dkmj.org

Texas A&M International University

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Boolean spaces

A Boolean space is a compact Hausdorff space with a clopen base.

A space is Boolean iff it is homeomorphic to a closed subspace of some 2^κ. But not every Boolean space is a retract of some 2^κ, and not every Boolean spaces is dyadic, that is, not even a continuous image of some 2^κ.

For a closed $C \subset 2^\kappa$, if we consider the coordinate projections $\pi_s \colon C \to 2^s$, $x \mapsto x \restriction s$, for finite $s \subset \kappa$, then we see that **a space is Boolean iff it is an inverse limit of surjections between finite discrete spaces.**

Special inverse limits

In short, a space is Boolean iff it is an (inverse) limit of **finite surjections** (between discrete spaces).

What if we place extra requirements on these finite surjections?

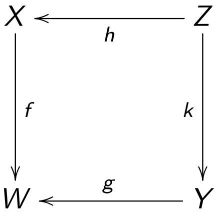
I am not aware of answers in the literature, except for one section of a book by Heindorf and Shapiro, *Nearly projective Boolean algebras*. (I'll say more about that later.)

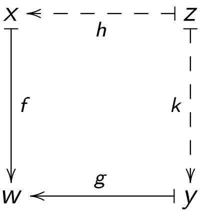
However, I have found a new characterization of the class of *retracts* of powers of 2 (also known as the Dugundji spaces or the absolute extensors of dimension zero (*i.e.*, AE(0) spaces)).

These retracts are exactly those limits of finite surjections that interact with each other in a "mutually surjective" way, roughly speaking.

Bicommutative squares

Bicommutative squares were used extensively by Ščepin in the 1970s. The concept apparently goes at least as far back as Kuratowski's 1966 *Topology, Vol. 1*.



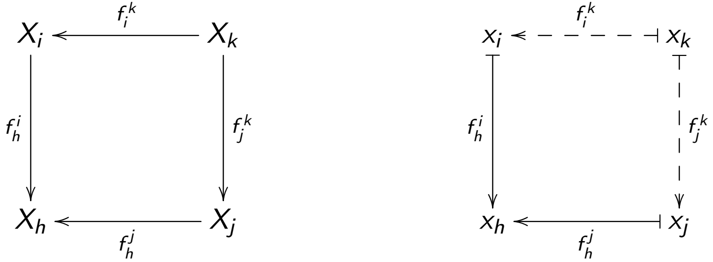


A commutative square of functions $W \xleftarrow[f]{} X \xleftarrow[h]{} Z \xrightarrow[k]{} Y \xrightarrow[g]{} W$ is said to bicommute if, for every pair (x, y) such that $f(x) = g(y)$, there exists z such that $(h, k)(z) = (x, y)$.

Bicommutative limits

Given an inverse limit system, say, $f_i^j \colon X_j \to X_i$ for all $i \leq j$ in some directed poset, every quadruple (h, i, j, k) with $h \leq i \leq k \geq j \geq h$ corresponds to a commutative square.

We can ask, does every such square bicommute?



Asking for squares with h arbitrarily low to bicommute is too much.
So, assume the poset is a meet-semilattice: $\forall i, j \; \exists i \wedge j = \inf\{i, j\}$.
If all the squares with $h = i \wedge j$ bicommute, say that the inverse limit X_∞ is a bicommutative limit of the X_i's.

Retracts of 2^{ω_1}

X is a retract of Y if there are continuous maps $X \xleftarrow[r]{} Y \xleftarrow[e]{} X$ such that $r \circ e = \mathrm{id}_X$.

Every second-countable Boolean space is a retract of 2^ω.

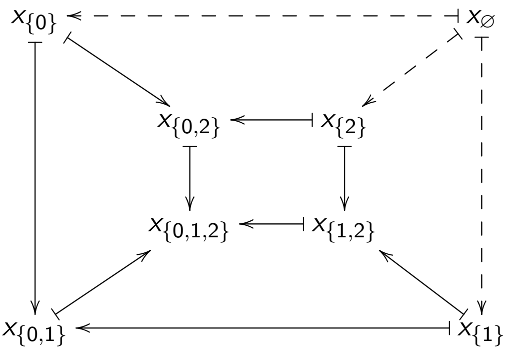
A space of uncountable weight λ is a retract of some 2^κ iff it is a retract of 2^λ.

Theorem (Heindorf and Shapiro). A Boolean space of weight $\aleph_1$ is a retract of 2^{ω_1} iff it is a bicommutative limit of finite surjections.

(Heindorf and Shapiro's result was actually stated and proved purely algebraically: a Boolean algebra of size $\aleph_1$ is projective iff it has the "strong Freese-Nation property.")

What if we consider *tricommutative* limits of finite surjections?

Tricommutative cubes



Given a commutative cube of functions $f_\sigma^\tau \colon X_\tau \to X_\sigma$ for $\tau \subset \sigma \subset 3$, we say it tricommutes if, for every triple $(x_{\{0\}}, x_{\{1\}}, x_{\{2\}})$ such that $f_{\{i,j\}}^{\{i\}}(x_{\{i\}}) = f_{\{i,j\}}^{\{j\}}(x_{\{j\}})$ for all $i < j < 3$, there exists $x_\varnothing$ such that $f_{\{i\}}^{\varnothing}(x_{\{i\}}) = x_\varnothing$ for all $i < 3$.

To define n-commuting n-cubes, replace 3 with n.

n-commutative limits

Given an inverse limit system, say, $f_i^j \colon X_j \to X_i$ for all $i \leq j$ in some directed meet-semilattice $(I, \leq)$ an n-cube subdiagram is a commutative n-cube of the form $f_{\varphi(\sigma)}^{\varphi(\tau)} \colon X_{\varphi(\tau)} \to X_{\varphi(\sigma)}$ for $\tau \subset \sigma \subset n$, for some $\wedge$-preserving map $\varphi \colon (\mathcal{P}(n), \supset) \to (I, \leq)$.

Say that an inverse limit system n-commutes if all its n-cube subdiagrams n-commute.

Say that an inverse limit system $(< \omega)$-commutes if, for all $n < \omega$, all its n-cube subdiagrams n-commute.

Retracts of 2^κ

Theorem. For a Boolean space X of weight $\aleph_n$, the following are equivalent.

- X is a retract of 2^{ω_n}
- X is an $(n+1)$-commutative limit of finite surjections.
- X is a $(<\omega)$-commutative limit of finite surjections.

Theorem. For a Boolean space X of weight $\lambda \geq \aleph_\omega$, the following are equivalent.

- X is a retract of 2^λ
- X is a $(<\omega)$-commutative limit of finite surjections.
- For each $n < \omega$, X is an n-commutative limit of finite surjections.

(These theorems are proved in my July 2016 arXiv preprint.)

The above theorems do not change if we require that the inverse limit system be indexed by a directed meet-semilattice instead of merely a directed poset.

A purely finite application

Theorem (Shapiro). The Vietoris hyperspace $\exp(2^{\omega_2})$ of nonempty closed subsets of 2^{ω_2} is not a continuous image (and, hence, not a retract of) 2^{ω_2}.

Corollary. Although 2^{ω_2} is a tricommutative inverse limit of finite surjections, $\exp(2^{\omega_2})$ is not.

Corollary. There is a tricommutative cube of finite surjections that is not tricommutative after after applying the exp functor.

Given $f\colon X \to Y$ continuous, $\exp(f)\colon \exp(X) \to \exp(Y)$ where $\exp(f)(A) = f[A]$ for all closed nonempty $A \subset X$.

In contrast, Heindorf and Shapiro proved algebraically that the exp functor preserves bicommutative squares of Boolean spaces. (And it's easier to prove this topologically for all spaces.)

Montemayor's tricommutative cube

Again, there is a tricommutative cube of finite surjections that is not tricommutative after after applying the exp functor.

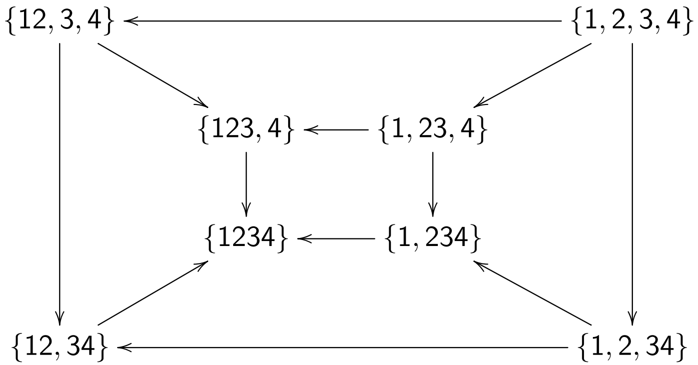
Though my non-constructive proof gave no bound, my guess was that a small example existed. I gave one of my students, René Montemayor, the task of finding an explicit example cube, actually, the Stone dual of such a cube, since I taught him the relevant background in terms of Boolean algebras.

In the algebraic formulation, one does not have to search for 8 sets and 12 surjections between them, but rather for 1 Boolean algebra and 3 subalgebras. The intersections of these subalgebras form the other 4 vertices of the cube; the inclusion maps form the 12 edges of the cube.

I suspected some cube with 2^3 at the top would work. René found that $2^2 \cong 4$, and no less, is enough.

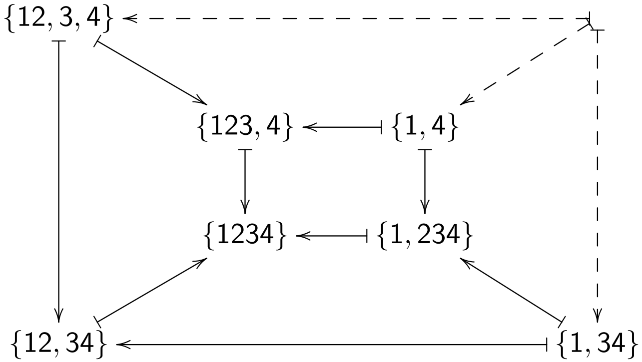
Montemayor's tricommutative cube

The following is the Stone dual of Montemayor's minimal example, before the exp functor is applied.



Tricommutativity destroyed

After applying the exp functor, the resulting cube's non-tricommutativity is witnessed below.



Why is tricommutativity destroyed?

Why does exp destroy tricommutativity but not bicommutativity?

From an algebraic perspective, the obstruction to exp preserving tricommutativity is that if $e_i \colon A_i \to B$ are inclusions of Boolean algebras for $i < 2$, then $\bigcup_{i<2} A_i$ could generate B yet $\bigcup_{i<2} \exp(e_i)[\exp(A_i)]$ not generate $\exp(B)$.

Subalgebras of the form $\langle (A_0 \cap A_2) \cup (A_1 \cap A_2) \rangle$ matter for tricommutativity but not for bicommutativity.

About the proofs

To prove that a retract R of 2^κ is an $(<\omega)$-commutative limit of finite surjections:

Use Koppelberg's characterization of retracts of powers of 2 as limits of continuous linear inverse systems of **open** surjections $f_\alpha^\beta \colon X_\beta \to X_\alpha$ where each $f_\alpha^{\alpha+1}$ extends to the first coordinate projection on $X_\alpha \times 2$.

The rest is algebra and simple transfinite recursion.

For the proof of the converse:

Use Haydon's characterization of retracts of powers of 2 as limits of continuous linear inverse systems of **open** surjections $f_\alpha^\beta \colon X_\beta \to X_\alpha$ where each $f_\alpha^{\alpha+1}$ extends to the first coordinate projection on $X_\alpha \times 2^\omega$.

More about the proofs

To prove an $(<\omega)$-commutative limit L of finite surjections $f_s^t \colon L_t \to L_s$ is a retract of 2^κ, and that $(n+1)$-commutative is enough if $w(L) \leq \aleph_n$:

1. Use a Davies sequence, that is, a sequence of countable $M_\alpha \prec (H(\theta), \in, \lhd)$ such that $(M_\beta)_{\beta<\alpha} \in M_\alpha$.
2. Each $\bigcup_{\beta<\alpha} M_\beta$ is a **finite** union $\bigcup_{i<k} N_i$ where $N_i \prec H(\theta)$.
3. If $\alpha \leq \omega_n$, then "finite" improves to "$\leq n$."
4. For a set E, define L/E by $x/E \neq y/E$ iff Boolean combinations of clopen sets in E separate x and y.
5. $(k+1)$-commutativity of $(L_\bullet, f_\bullet^\bullet)$ implies that the natural quotient map $L/M_\alpha \to L/(M_\alpha \cap \bigcup_{i<k} N_i)$ is **open**.
6. By elementarity, $L/\bigcup_{i<k} N_i$, which equals $L/\bigcup_{\beta<\alpha} M_\beta$, is an open quotient of $L/\bigcup_{\beta<\alpha+1} M_\beta$.
7. By Haydon's criterion, L is a retract of $2^{w(L)}$.

Questions

1. Is there a tricommutative limit of finite surjections that is not a retract of some 2^κ?
2. Is there a tricommutative limit of open surjections between second countable Boolean spaces that is not a tricommutative limit of finite surjections? (The answer is 'yes' if tri- is replaced by bi-.)
3. Is there a tricommutative limit of open surjections between second countable Boolean spaces that is not dyadic?

Any example answering (1), (2), or (3) must have weight $\geq \aleph_3$.

For inverse limit systems of open surjections between second countable Boolean spaces, I can cook up an n-commutative system whose limit is not the limit of any $(n+1)$-commutative system (using Davies sequences, of course). But I don't know how to do this for finite surjections when $n \geq 3$.

I know (1) or (2) has a 'yes' answer, but I don't know which.