Tukey classes of local bases in compacta

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16th Boise Extravaganza in Set Theory

Motivation

- Study homeomorphism-invariant local properties of compacta in hopes of obtaining negative results about open questions about homogeneous compacta.
- Specifically, study order-theoretic properties of local bases of compacta.

Topological preliminaries

- **Definition.** A *local base* at a point p in a space X is a family $\mathcal{F}$ of open neighborhoods of p such that every neighborhood of p contains an element of $\mathcal{F}$.

- **Definition.** A *local* π*-base* at a point p in a space X is a family $\mathcal{F}$ of nonempty open subsets of X such that every neighborhood of p contains an element of $\mathcal{F}$.

- **Definition.** $\chi(p, X) = \min\{|\mathcal{F}| : \mathcal{F}$ local base at $p\}$.

- **Definition.** $\pi\chi(p, X) = \min\{|\mathcal{F}| : \mathcal{F}$ local π-base at $p\}$.

Tukey equivalence

- **Definition.** A directed set P is Tukey reducible to a directed set Q (written $P \leq_T Q$) if there is map from P to Q such that the image of every unbounded set is unbounded.

- **Theorem** (Tukey, 1940). $P \equiv_T Q$ iff P and Q embed as cofinal subsets of a common third directed set.

- **Convention.** Families of open sets are ordered by $\supseteq$.

- **Corollary.** Every two local bases at a common point are Tukey equivalent.

- $P \leq_T Q \Rightarrow \mathsf{cf}(P) \leq \mathsf{cf}(Q)$
- $\alpha \leq_T \beta \Leftrightarrow \mathsf{cf}(\alpha) = \mathsf{cf}(\beta)$
- $P \leq_T P \times Q$
- If $P \leq_T R \geq_T Q$, then $P \times Q \leq_T R$.
- **Convention.** Sets of the form $[A]^{<\kappa}$ are ordered by $\subseteq$.
- $P \leq_T [\mathsf{cf}(P)]^{<\omega}$
- $[A]^{<\omega} \leq_T [B]^{<\omega} \Leftrightarrow |A| \leq |B| + \omega$

- **Theorem 1.** Let X be a compactum and $\kappa = \min_{p \in X} \pi\chi(p, X)$. Then there is a local base $\mathcal{F}$ at some point in X such that $[\kappa]^{<\omega} \leq_T \mathcal{F}$.

- **Corollary.** Let X be a compactum such that every point has a local base with no uncountable antichains (in the sense of incomparability). Then there is a countable local π-base at some point in X.

- **Proof.** Use $\omega_1 \to (\omega_1, \omega + 1)$ to conclude that $[\omega_1]^{<\omega}$ is not Tukey reducible to any local base of X. Apply Theorem 1.

- **Definition.** A directed set P is *flat* if $P \equiv_T [\mathrm{cf}(P)]^{<\omega}$. A point in a space is flat if it has a flat local base.

- **Corollary.** Let X be a compactum such that $\pi\chi(p, X) = \chi(q, X)$ for all $p, q \in X$. Then X has a flat point.

- **Definition.** A compactum is *dyadic* if it is a continuous image of a power of 2.

- **Theorem 2.** Every point in every dyadic compactum is flat.

- **Question.** Is every point in every homogeneous compactum flat?

Independence results about $\beta\omega \setminus \omega$

- **Theorem 3** (Dow & Zhou, 1999). There is a flat point in $\beta\omega \setminus \omega$.

- **Question.** Is it consistent that all points in $\beta\omega \setminus \omega$ are flat?

- **Theorem 4** (MA). If $\omega \leq \mathrm{cf}(\kappa) = \kappa \leq \mathfrak{c}$, then $\beta\omega \setminus \omega$ has a local base Tukey equivalent to $[\mathfrak{c}]^{<\kappa}$.

- **Question.** Assuming MA, does Theorem 4 enumerate all Tukey classes of local bases of $\beta\omega \setminus \omega$?

- **Definition.** The *pseudointersection number* $\mathfrak{p}$ is the least κ for which MA_κ fails for some σ-centered poset.

- **Theorem 5.** If κ is a regular infinite cardinal less than $\mathfrak{p}$ and Q is a κ-directed set, then no local base in $\beta\omega \setminus \omega$ is Tukey equivalent to $\kappa \times Q$. **4/9/2007: The second κ should be a κ^+.**

- **Corollary** (MA). If κ and λ are distinct regular infinite cardinals, then no local base in $\beta\omega \setminus \omega$ is Tukey equivalent to $\kappa \times \lambda$.

- **Theorem 6.** Given any two regular uncountable cardinals κ and λ, it is consistent with ZFC that $\beta\omega \setminus \omega$ has a local base Tukey equivalent to $\kappa \times \lambda$.

- **Remark.** It is not hard to show that, for a fixed κ, a construction of Brendle and Shelah (1999) can be trivially modified to yield of a model of ZFC in which $\beta\omega \setminus \omega$ has a local base Tukey equivalent to $\kappa \times \lambda$ for each λ in an arbitrary set of regular cardinals exceeding κ.

References

J. Brendle and S. Shelah, *Ultrafilters on ω—their ideals and their cardinal characteristics*, Trans. AMS **351** (1999), 2643–2674.

A. Dow and J. Zhou, *Two real ultrafilters on ω*, Topology Appl. **97** (1999), no. 1–2, 149–154.

J. W. Tukey, *Convergence and uniformity in topology*, Ann. of Math. Studies, no. 2, Princeton Univ. Press, Princeton, N. J., 1940.

- **About the proof of Theorem 2.** It suffices to build a local base $\mathcal{F}$ at a given point such that $\mathcal{F}$ is ω-like (*i.e.*, all bounded sets are finite). We proceed by induction on the weight of the space, using a chain of elementary substructures of some H_θ and a nice reflection property of free boolean algrebras, which are the Stone duals of powers of 2.

- **About the proof of Theorem 1.** It suffices to find a κ-sized family of neighborhoods of some point p such that the intersection of an infinite subfamily never has p in its interior. Given a family $\mathcal{F}$ of sets, set $\Phi(\mathcal{F}) = \Big\{\langle\sigma, \langle E_i\rangle_{i<n}\rangle \in [\mathcal{F}]^{<\omega} \times ([\mathcal{F}]^{\omega})^{<\omega} : \forall\tau \in \prod_{i<n} E_i \quad \bigcap\sigma \subseteq \overline{\bigcup \mathrm{ran}(\tau)}\Big\}$. The trick is to iteratively construct open neighborhoods $\langle U_\alpha\rangle_{\alpha<\kappa}$ of a common point such that $\Phi(\{U_\alpha\}_{\alpha<\kappa}) = \emptyset$.

- **About the proof of Theorem 4.** Use Solovay's Lemma to iteratively build a local base $\mathcal{F}$ at a P_κ-point that also satisfies $\Phi_\kappa(\mathcal{F}) = \emptyset$ where $\Phi_\kappa(\mathcal{F})$ is $\Phi(\mathcal{F})$ with $[\mathcal{F}]^{\omega}$ replaced by $[\mathcal{F}]^{\kappa}$.