Box products and singular cardinals

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June 1, 2011
BLAST at Lawrence, KS

Acknowledgements

This is joint work with Menachem Kojman and Santi Spadaro.

Convention

All spaces are T_3 (regular and Hausdorff).

Some order properties

Definition

- A preorder P is **strongly κ-Noetherian** if every subset of size κ lacks a lower bound.
- A preorder P is **strongly κ-Artinian** if every subset of size κ lacks an upper bound.

Convention

Assume sets are ordered by $\subseteq$ unless stated otherwise.

Examples

- A base $\mathcal{B}$ of a topological space is strongly κ-Noetherian if and only if, for every subset $\mathcal{A}$ of $\mathcal{B}$ of size κ, $\bigcap \mathcal{A}$ has empty interior.
- The canonical base of 2^λ—the finitely supported boxes—is strongly $\aleph_0$-Noetherian because a finite function has only finitely many subfunctions.

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- The canonical base of 2^λ—the finitely supported boxes—is strongly $\aleph_0$-Noetherian because a finite function has only finitely many subfunctions.
- X_δ is X with all G_δ-sets declared open.
- The canonical base of 2^λ_δ—the countably supported boxes—is strongly $\left(2^{\aleph_0}\right)^+$-Noetherian (when ordered by $\subseteq$) because a countable function has at most $2^{\aleph_0}$-many subfunctions.

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- (Malykhin) If $\mathcal{B}$ is a base of a space X, then $X^{|\mathcal{B}|}$ has a strongly $\aleph_0$-Noetherian base.

An incomplete analogy

Cellularity

- The cellularity $c(X)$ of X is the supremum of the sizes of pairwise disjoint families of open subsets of X.
- (Juhász) If X is countably compact, then $c(X_\delta) \leq 2^{c(X)}$.

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Noetherian type

- The Noetherian type $\mathrm{Nt}(X)$ of X is the least infinite cardinal κ such that X has a strongly κ-Noetherian base.
- Assume GCH. If X is countably compact and cf(ntX) is uncountable, then $\mathrm{Nt}(X_\delta) \leq 2^{\mathrm{Nt}(X)}$.
- What if we drop GCH? What if cf$(\mathrm{Nt}(X)) = \omega$?

If we drop GCH

We can get by with weaker versions of GCH.

- If X is countably compact, $\mathrm{cf}(\mathrm{Nt}(X)) > \omega$, and $\lambda^{\aleph_0} \leq \mathrm{Nt}(X)$ for all $\lambda < \mathrm{Nt}(X)$, then $\mathrm{Nt}(X_\delta) \leq 2^{\mathrm{Nt}(X)}$.
- Similarly, if X is countably compact, $\mathrm{Nt}(X) \leq \kappa$, and $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$, then $\mathrm{Nt}(X_\delta) \leq \kappa$.

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Some cardinal arithmetics lead to counterexamples.

- Let X be the one-point compactification of the discrete space of size $\aleph_\omega$.
- (Gitik-Magidor) $2^{\aleph_0} < \aleph_\omega^{\aleph_0} = 2^{\aleph_{\omega+1}} = \aleph_{\omega+2}$ is consistent relative to a measurable κ with $o(\kappa) = \kappa^{++}$.
- Assuming the above cardinal arithmetic, $\mathrm{Nt}(X) = \aleph_{\omega+1}$ and $\mathrm{Nt}(X_\delta) = \aleph_{\omega+3} > 2^{\mathrm{Nt}(X)}$.

If $\mathrm{cf}(\mathrm{Nt}(X)) = \omega$

- For simplicity, suppose X is of the form 2^λ, which implies $\mathrm{Nt}(X) = \aleph_0$.
- $2_\delta^{\aleph_n}$ has an $\aleph_1$-strongly Noetherian base, for all $n < \omega$.
- Does $2_\delta^{\aleph_\omega}$ have an $\aleph_1$-strongly Noetherian base?

Combinatorially speaking

- $\mathrm{Nt}\left(2^{\lambda}_{\delta}\right) \leq \kappa$ if and only if $[\lambda]^{\aleph_0}$ has a strongly κ-Artinian cofinal subset.
- A subset $\mathcal{F}$ of $[\lambda]^{\aleph_0}$ is strongly κ-Artinian if and only if $|\bigcup \mathcal{A}| \geq \aleph_1$ for all $\mathcal{A} \in [\mathcal{F}]^{\kappa}$.

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- For each $n < \omega$, $[\aleph_n]^{\aleph_0}$ has a strongly $\aleph_1$-Artinian cofinal subset.
- What about $[\aleph_\omega]^{\aleph_0}$?

ZFC upper bounds on $\mathrm{Nt}\left(2_{\delta}^{\aleph_\omega}\right)$

- (Easy) $\mathrm{Nt}\left(2_{\delta}^{\aleph_\omega}\right) \leq \left(2^{\aleph_0}\right)^{+}$.

ZFC upper bounds on $\mathrm{Nt}\left(2_\delta^{\aleph_\omega}\right)$

- (Easy) $\mathrm{Nt}\left(2_\delta^{\aleph_\omega}\right) \leq \left(2^{\aleph_0}\right)^+$.
- $\mathrm{Nt}\left(2_\delta^{\aleph_\omega}\right) \leq \aleph_4$.
- The proof uses a (max-pcf) scale of $\prod_{n\in\omega} \aleph_n$ modulo an ideal on ω, and club guessing on $\{\alpha < \omega_3 : \mathrm{cf}(\alpha) = \omega_1\}$.

Conditional upper bounds

- $S^\lambda_\kappa = \{i < \lambda : \mathrm{cf}(i) = \kappa\}$.
- The approachability ideal $I[\lambda]$ consists of the sets $S \subseteq \lambda$ for which there is a club $E \subseteq \lambda$ and a sequence $\overline{C}$ such that
  - C_i is a cofinal subset of i for all $i < \lambda$.
  - C_i has order type $\mathrm{cf}(i)$ for for all $i \in E$.
  - $\{C_i \cap j : j < i\} \subseteq \{C_j : j < i\}$ for all $i \in S \cap E$.

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- If $\lambda = \mathrm{cf}\left([\aleph_\omega]^{\aleph_0}\right)$ and $S^\lambda_\kappa \in I[\lambda]$, then $\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) \leq \kappa$.
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- Hence, if $\square_{\aleph_\omega}$ and $\mathrm{cf}\left([\aleph_\omega]^{\aleph_0}\right) = \aleph_{\omega+1}$, then $\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) = \aleph_1$.

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- Hence, if $\square_{\aleph_\omega}$ and $\mathrm{cf}\left([\aleph_\omega]^{\aleph_0}\right) = \aleph_{\omega+1}$, then $\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) = \aleph_1$.
- (Sharon-Viale) MM implies $S^{\aleph_{\omega+1}}_{\omega_2} \in I[\aleph_{\omega+1}]$.
- Therefore, MM implies $\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) \leq \aleph_2$.

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- $(\lambda, \kappa) \twoheadrightarrow (\lambda', \kappa')$ means every structure $\mathcal{A}$ with universe λ and countable signature has a substructure $\mathcal{B}$ of cardinality λ' such that $|\kappa \cap \mathcal{B}| = \kappa'$.
- (Levinski-Magidor-Shelah) $\mathrm{CC}_{\aleph_\omega}$, by which we mean $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, is consistent with ZFC+GCH, relative to a 2-huge cardinal.
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- (Soukup) $\mathsf{CC}_{\aleph_\omega}$ implies $\mathrm{Nt}\left(2_{\delta}^{\aleph_\omega}\right) \geq \aleph_2$.
- More generally, $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_{n-1}, \aleph_{n-2})$ implies $\mathrm{Nt}\left(2_{\delta}^{\aleph_\omega}\right) \geq \aleph_n$.
- We don't know if it is consistent to have $n = 3$ or $n = 4$.

π-bases

- $\pi\mathrm{Nt}(X)$ is the least κ such that X has a strongly κ-Noetherian π-base.
- $\pi\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) \leq \kappa$ if and only if there is a strongly κ-Artinian cofinal family of countable partial functions from $\aleph_\omega$ to 2.

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- If $2^{\aleph_0} > \aleph_\omega$, then $\pi\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) = \aleph_1$.
- $\mathsf{CC}_{\aleph_\omega}$ is consistent with $2^{\aleph_0} > \aleph_\omega$ because ccc forcings preserve $\mathsf{CC}_{\aleph_\omega}$.

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- $\mathrm{CC}_{\aleph_\omega}$ is consistent with $2^{\aleph_0} > \aleph_\omega$ because ccc forcings preserve $\mathrm{CC}_{\aleph_\omega}$.
- $\mathrm{CC}_{\aleph_\omega}$ and $2^{\aleph_0} < \aleph_\omega < 2^{<\aleph_\omega}$ together imply $\pi\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right) \geq \aleph_2$.
- Is $\mathrm{CC}_{\aleph_\omega}$ consistent with $2^{\aleph_0} < \aleph_\omega < 2^{<\aleph_\omega}$?
- Adding $\aleph_\omega$ or more Cohen subsets of ω_1 destroys $\mathrm{CC}_{\aleph_\omega}$.

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- Adding $\aleph_\omega$ or more Cohen subsets of ω_1 destroys $\mathrm{CC}_{\aleph_\omega}$.
- What happens to $\pi\mathrm{Nt}\left(2^{\aleph_\omega}_\delta\right)$ in models of $\mathrm{CC}_{\aleph_\omega}$ + GCH?