Higher-order amalgamation of algebraic structures, Part I

David Milovich
http://dkmj.org

Welkin Sciences
Colorado Springs, CO

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Overlapping structures

Definition

Given a class $\mathcal{V}$ of structures, say that structures $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$ with underlying sets A, B overlap in $\mathcal{V}$ if $\mathfrak{A}$ and $\mathfrak{B}$ have a common substructure $\mathfrak{C} \in \mathcal{V}$ with underlying set $A \cap B$.

Example

$\mathbb{R}[x]/(x^2+1)$ and $\mathbb{R}[y]/(y^2+1)$ are algebraically closed fields whose intersection is a common subfield, $\mathbb{R}$. They overlap in the class of fields but not in the class of algebraically closed fields.

Definition

$\mathfrak{A} \cap \mathfrak{B}$ denotes the common substructure of $\mathfrak{A}$ and $\mathfrak{B}$ with universe $A \cap B$, if it exists.

Amalgamation of structures

Definition

Say that a set $\mathcal{S}$ of overlapping structures amalgamates in $\mathcal{V}$ if there is structure $D \in \mathcal{V}$ such that every $A \in \mathcal{S}$ is a substructure of D. Call any such D an amalgamation of $\mathcal{S}$ in $\mathcal{V}$.

Example

• $\mathbb{R}[x]/(x^2+1)$ and $\mathbb{R}[y]/(y^2+1)$ amalgamate in the class of commutative rings because distinct elements of

$$\big(\mathbb{R}[x]/(x^2+1)\big) \cup \big(\mathbb{R}[y]/(y^2+1)\big)$$

remain distinct in the ring $\mathbb{R}[x,y]/(x^2+1, y^2+1)$.
• $\mathbb{R}[x]/(x^2+1)$ and $\mathbb{R}[y]/(y^2+1)$ do NOT amalgamate in the class of integral domains because

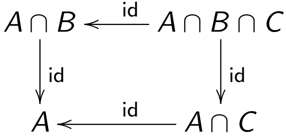
$$x^2+1 \equiv 0 \equiv y^2+1 \Rightarrow (x+y)(x-y) = (x^2+1)-(y^2+1) \equiv 0.$$

• $\mathbb{R}[x]/(x^2+1)$ and $\mathbb{R}[y]/(y^2+1)$ do amalgamate in the class of skew fields: declare $yx = -xy$ to get the quaternions.

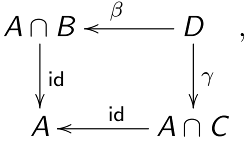
How to abstract “overlap” to category theory

- To abstract two overlapping structures, replace the two instances of the substructure relation with monomorphisms, which are usually just injective homomorphisms.
- But to abstract three overlapping structures requires more:

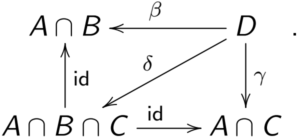
Main idea: The diagram to the right is not just a commutative square; it's also a <u>pullback square</u>.



That is, for any commutative square



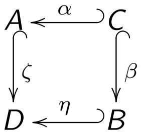
β and γ factor through inclusions of $A \cap B \cap C$ via a unique homomorphism δ to $A \cap B \cap C$.



To abstract n overlapping structures, use “pullback hypercubes.”

How to abstract amalgamation to category theory

- An amalgamation D of overlapping structures A, B with $A \cap B = C$ abstracts to a pullback square of monomorphisms.

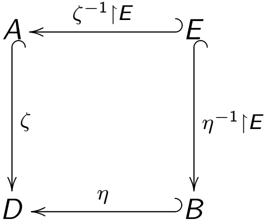


- Why not just a commutative square of monomorphisms?

Requiring a pullback square abstracts the requirement that if $x \in A$, $y \in B$, and $x, y \notin A \cap B$, then x and y remain distinct in D.

If $\zeta \circ \alpha = \eta \circ \beta$ and ζ and η were injective but $E = \zeta A \cap \eta B \not\subset \zeta\alpha C$,

then the square to the right would commute but $\zeta^{-1} \restriction E$ could not be of the form $\alpha \circ \delta$ and, hence, ζ and η do not factor through α and β via any $\delta \colon E \to C$.



Compactness

Lemma

Suppose that:

- $A_i, A_i \cap A_j \in \mathcal{V}$ *for all* $i, j \in E$.
- $\mathcal{V}$ *is axiomatized by a set of first-order formulas.*
- $\{B_i : i \in F\}$ *amalgamates in* $\mathcal{V}$ *for all finite* $F \subset E$ *and all finitely generated substructures* $B_i \subset A_i$ *with* $B_i \in \mathcal{V}$.

Then $\{A_i : i \in E\}$ *amalgamates in* $\mathcal{V}$.

Proof.

- Let $\mathbb{P}$ denote the set of $B = (B_i : i \in F)$ as above.
- Partially order $\mathbb{P}$ by $B \leq B'$ iff $F \subset F'$ and $B_i \subset B'_i$ for all i.
- Let $D_{\mathcal{U}}$ be the ultraproduct $\prod_B D_B/\mathcal{U}$ where $\mathcal{U}$ is an ultrafilter on $\mathbb{P}$ such that $\{B' \mid B' \geq B\} \in \mathcal{U}$ for all B.
- Then the diagonal map $x \mapsto (B \mapsto x : x \in \bigcup_i B_i)/\mathcal{U}$ defines an injection from $\bigcup_i A_i$ to $D_{\mathcal{U}}$ that restricts to a homomorphism on each A_i. □

Semigroup (and ring) amalgamation is hard

- (Kimura, 1957) There are overlapping finite commutative semigroups S_1, S_2 that do not amalgamate as semigroups.
- It follows that there are two finite commutative rings that do not amalgamate as rings.
- (Sapir, 1997; Jackson, 2000) There is no algorithm that can decide whether two arbitrary finite semigroups amalgamate. Likewise for finite rings.

Example

Define overlapping commutative semigroups $S_i = \{0, a, b, c_i\}$ for $i = 1, 2$ as shown. If some semigroup T contained $S_1 \cup S_2$, then we would reach the contradiction

$$a = c_1 b = c_1(ac_2) = (c_1 a)c_2 = ac_2 = b.$$

$\cdot$	c_2	0	a	b	c_1
c_2	c_2	0	b	b	
0	0	0	0	0	0
a	b	0	0	0	a
b	b	0	0	0	a
c_1		0	a	a	c_1

Δ-systems of groups

Call a set $\mathcal{S}$ of overlapping structures a $\underline{\Delta\text{-system}}$ (or sunflower) in $\mathcal{V}$ if $\mathcal{S} \subset \mathcal{V}$ and there is a root $R \in \mathcal{V}$ such that $A \cap B = R$ for all distinct $A, B \in \mathcal{S}$.

Theorem (Schreier, 1927)

Every Δ-system of groups amalgamates in the class of groups.

Corollary

Any two overlapping groups amalgamate in the class of groups.

Schreier's Theorem remains true if we replace "groups" with "abelian groups." And the proof is much easier!

Δ-systems of abelian groups (or left R-modules)

- For $n > 2$, use induction:
  1. Assume a Δ-system of abelian groups $G_1, \ldots, G_n$ with root H.
  2. Assume K is an amalgamation of $G_1, \ldots, G_{n-1}$.
  3. Replace K with an isomorphic copy K' such that $K' \cap G_n = H$.
  4. Amalgamate K' and G_n.
- Base case of overlapping abelian groups $(A, +_A, 0), (B, +_B, 0)$:
  1. Choose isomorphisms $\alpha\colon A \to A'$, $\beta\colon B \to B'$ such that $A' \cap B' = \{0\}$.
  2. Let D be a direct sum $A' \oplus B' \supset A' \cup B'$ and $E = D/N$ where
  $$N = \{\alpha x - \beta x : x \in A \cap B\}.$$
  3. Check that $(\alpha/N) \cup (\beta/N)$ is injective:
  $$\begin{aligned} \alpha a/N = \beta b/N &\Rightarrow \alpha a - \beta b = (\alpha - \beta)x \\ &\Rightarrow 0 = \alpha(x - a) + \beta(b - x) \\ &\Rightarrow (0,0) = (x - a, b - x) \Rightarrow a = x = b \end{aligned}$$

Δ-systems of posets and of Boolean algebras

To amalgamate a Δ-system of posets $\{(P_i, \leq_i) : i \in E\}$ with root R, declare $x \leq y$ iff

- $x \leq_i y$ for some $i \in E$ or
- $x \leq_i z \leq_j y$ for some $i, j \in E$ and $z \in R$.

Claim: $\leq$ is transitive, antisymmetric, and such that $\leq \restriction P_i = \leq_i$.

A Δ-system of Boolean algebras amalgamates essentially the same way a Δ-system of abelian groups does. Base case:

1. Assume overlapping Boolean algebras A_1, A_2.
2. Let B be the coproduct $A_1 \oplus A_2$, which is the Boolean algebra generated by copies $e_i(a)$ of a for each i and $a \in A_i$, and the relations $e_i(0) = 0$, $e_i(-x) = -e_i(x)$, and $e_i(x \wedge y) = e_i(x) \wedge e_i(y)$ for each i and $x, y \in A_i$.
3. Let $C = B/I$ where I is the ideal
$$\left\{ b \wedge \bigvee_{i=1}^{n} (e_1(x_i) \wedge e_2(-x_i)) : b \in B,\, x_1, \ldots, x_n \in A_1 \cap A_2 \right\}.$$
4. Claim: $(e_1/I) \cup (e_2/I)$ is injective.

Beyond Δ-systems

Not every three overlapping posets A, B, C amalgamate:

$$x <_A y <_B z <_C x.$$

A Boolean algebra is a partially ordered by $x \leq y \Leftrightarrow x = x \wedge y$.
So, not every three overlapping Boolean algebras amalgamate, not even in the class of posets.

(H. Neumann) Higher amalgamation of groups is very interesting:

- (1948) There are three overlapping groups that do not amalgamate in the class of groups.
- (1951) But any three overlapping abelian groups amalgamate in the class of abelian groups.
- (1953, with B.H. Neumann) But there are four overlapping abelian groups that do not amalgamate in the class of groups.
- (1950) Yet, any set of overlapping locally cyclic groups amalgamates in the class of abelian groups.

Why three overlapping abelian groups amalgamate

- Suppose we have a three overlapping abelian groups A, B, C. (Our argument also works for three overlapping left R-modules.)

- Choose isomorphisms $\alpha\colon A \to A'$, $\beta\colon B \to B'$, $\gamma\colon C \to C'$ such that A', B', C' is a Δ-system with root $\{0\}$.

- Let $D = A' \oplus B' \oplus C' \supset A' \cup B' \cup C'$ and $E = D/N$ where

$$N = \{(\alpha - \beta)x + (\beta - \gamma)y + (\gamma - \alpha)z \\ : (x, y, z) \in (A \cap B) \times (B \cap C) \times (C \cap A)\}.$$

$$\begin{aligned}
& \boldsymbol{\alpha a/N = \beta b/N} \\
\Rightarrow \quad & \alpha a - \beta b = (\alpha - \beta)x + (\beta - \gamma)y + (\gamma - \alpha)z \\
\Rightarrow \quad & 0 = \alpha(x - z - a) + \beta(b - x + y) + \gamma(z - y) \\
\Rightarrow \quad & (0, 0, 0) = (x - z - a, b - x + y, z - y) \\
\Rightarrow \quad & \boldsymbol{a} = x - z = x - y = \boldsymbol{b}
\end{aligned}$$

- This proof doesn't work for $n > 3$ because then $\binom{n}{2} > n$.

Four non-amalgamating overlapping free abelian groups

Start with the following $G_1, G_2, G_3 \subset \langle a, b, c \rangle$.

$$(G_1, +_1) = \langle a, b \rangle \qquad (G_2, +_2) = \langle a, c \rangle \qquad (G_3, +_3) = \langle b, c \rangle$$

$$G_1 \cap G_2 = \langle a \rangle \qquad G_1 \cap G_3 = \langle b \rangle \qquad G_2 \cap G_3 = \langle c \rangle$$

Then choose $(G_4, +_4) = \langle a +_1 b, a +_2 c \rangle$
such that for all nonzero integers m, n,

- G_1 and G_4 agree on what $m(a + b)$ is, and
- G_2 and G_4 agree on what $n(a + c)$ is, but
- $[m(a + b)] +_4 [n(a + c)] \notin G_1 \cup G_2 \cup G_3$.

Then G_1, G_2, G_3, G_4 overlap but do not amalgamate in the class of abelian groups because $\boxplus_{i=1}^{4} G_i$ equates

$$b -_3 c \qquad \text{and} \qquad (a +_1 b) -_4 (a +_2 c).$$

This example generalizes to left R-modules.

Pushout squares

• In an equationally axiomatized class of structures, a coproduct $\bigoplus_{i \in E} A_i$ of structures $(A_i : i \in E)$ is a structure generated by

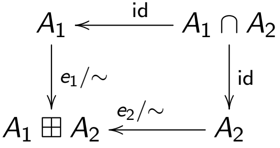
- elements $e_i(x)$ for each $i \in E$ and $x \in A_i$,
- relations saying that each e_i is an isomorphism, and
- relations saying that $\bigoplus_{i \in E} A_i$ satisfies the axioms of $\mathcal{V}$.

• All our binary amalgamation proofs start the same way: take a coproduct $e_1(A_1) \oplus e_2(A_2)$ of copies of A_1 and A_2 and then take the quotient

$$A_1 \boxplus A_2 = (e_1(A_1) \oplus e_2(A_2))/{\sim}$$

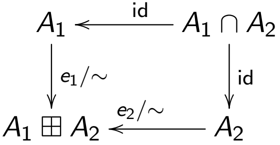
where $\sim$ is generated by $e_1(x) = e_2(x)$ for $x \in A_1 \cap A_2$.

• Even if A_1 and A_2 do not amalgamate, $A_1 \boxplus A_2$ is special: the diagram to the right is a pushout square...

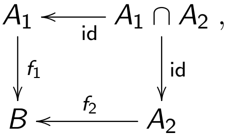


Pushout squares (continued)

The diagram to the right is a pushout square.



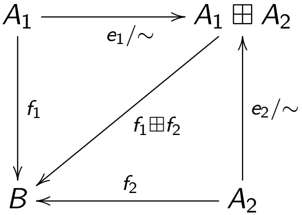
That is, for any commutative square



f_1 and f_2 factor through $e_1/\sim$ and $e_2/\sim$ via a unique homomorphism

$$f_1 \boxplus f_2 \colon A_1 \boxplus A_2 \to B$$

called the mediating morphism.



Therefore, if $f_1 \cup f_2$ is injective, then so is $(e_1/\sim) \cup (e_2/\sim)$.
Therefore, B is an amalgamation of A_1, A_2 if and only if an isomorphic copy of $A_1 \boxplus A_2$ is.

Pushout hypercubes

• More generally, define a (generalized) pushout $\boxplus_{i \in E} A_i$ to be an isomorphic copy of a quotient of a coproduct, of the form

$$\theta\left(\left[\bigoplus_{i \in E} e_i(A_i)\right] /\!\sim\right)$$

where θ and each e_i are isomorphisms and $\sim$ is generated by $e_i(x) = e_j(x)$ for all $i, j \in E$ and $x \in A_i \cap A_j$.

• Properly speaking, a pushout consists of $\boxplus_{i \in E} A_i$ and its coprojections

$$\boxplus_i = \theta \circ (e_i/\!\sim) \colon A_i \to \boxplus_{i \in E} A_i.$$

• Pushouts makes sense in any class of structures where coproducts and quotients make sense (*i.e.*, in any category with colimits).

• Overlapping structures amalgamate if and only if they are amalgamated by a pushout, provided pushouts are well-defined.

Linear amalgamation

Between pushouts $\boxplus_{i=1}^{m} A_i$ with coprojections $\boxplus_i^{(m)}$ for $1 \leq i \leq m \leq n$, there are unique mediating morphisms:

$$A_1 \xrightarrow{\boxplus_{\leq 1}} \boxplus_{i=1}^{2} A_i \xrightarrow{\boxplus_{\leq 2}} \boxplus_{i=1}^{3} A_i \xrightarrow{\boxplus_{\leq 3}} \cdots \xrightarrow{\boxplus_{\leq n-1}} \boxplus_{i=1}^{n} A_i$$

$$\begin{array}{ccc} A_i & \xrightarrow{\boxplus_i^{(m)}} & \boxplus_{i=1}^{m} A_i \\ & {\scriptstyle \boxplus_i^{(m+1)}} \searrow & \downarrow {\scriptstyle \boxplus_{\leq m}} \\ & & \boxplus_{i=1}^{m+1} A_i \end{array}$$

Definition

In a class with well-defined pushouts, say that overlapping structures $A_1, \ldots, A_n$ linearly amalgamate if there exists pushouts $\boxplus_{i=1}^{m} A_i$ for $m = 1, \ldots, n$ such that $\bigcup_{i=1}^{n} \boxplus_i^{(n)}$ and each $\boxplus_{\leq m}$ are indentity maps.

$$A_1 \subset \boxplus_{i=1}^{2} A_i \subset \boxplus_{i=1}^{3} A_i \subset \cdots \subset \boxplus_{i=1}^{n} A_i \supset \bigcup_{i=1}^{n} A_i$$

Iterated binary amalgamation

• If $(\boxplus_{i=1}^{m} A_i) \boxplus A_{m+1}$ exists, then it is $\boxplus_{i=1}^{m+1} A_i$.

(That is, any structure satisfying the definition of $(\boxplus_{i=1}^{m} A_i) \boxplus A_{m+1}$ must also satisfy the definition of $\boxplus_{i=1}^{m+1} A_i$.)

• If $A_1, \ldots, A_n$ linearly amalgamate, then $(\boxplus_{i=1}^{m} A_i) \boxplus A_{m+1}$ exists for all $m < n$.

• If $\mathcal{V}$ has well-defined pushouts and $A_1, \ldots, A_n$ amalgamate in $\mathcal{V}$ but do not linearly amalgamate in $\mathcal{V}$,

then some $(\boxplus_{i=1}^{m} A_i) \boxplus A_{m+1}$ does not exist

because A_{m+1} and $\boxplus_{i=1}^{m} A_i$ are not overlapping structures.

Nonlinear amalgamation

Example

Consider the following overlapping abelian groups.
(This example also works for left R-modules.)

$$A_1 = \langle x \rangle \qquad A_2 = \langle y \rangle \qquad A_3 = \langle z \rangle$$

$$A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \{0\}$$

$$A_4 = \langle x, y, z;\ x + y + z = 0 \rangle \supset A_1 \cup A_2 \cup A_3$$

Trivially, A_4 amalgamates A_1, A_2, A_3, A_4. And $A_4 = \boxplus_{i=1}^{4} A_i$.

But $x + y + z \neq 0$ in $\boxplus_{i=1}^{3} A_i = \bigoplus_{i=1}^{3} A_i = \langle x, y, z \rangle$.

Hence, $\boxplus_{i=1}^{3} A_i$ and A_4 disagree about $x + y + z$.

Thus, $\left(\boxplus_{i=1}^{3} A_i\right) \boxplus A_4$ does not exist and $\boxplus_{i=1}^{3} A_i \not\subset \boxplus_{i=1}^{4} A_i$.

The direct summand property

Definition

Say that a class $\mathcal{V}$ has the direct summand property if

- binary coproducts in $\mathcal{V}$ are well-defined and,
- for every $A, B \in \mathcal{V}$ with A a substructure of B, there exists $C \in \mathcal{V}$ such that $A \oplus C = B$.

Example

The following classes have the direct summand property.

- The class of vector spaces over a fixed field.
- The class of divisible groups.

The direct summand property applied

Definition

In a class $\mathcal{V}$, if A, B overlap, then let $A \ominus B$ denote some $C \in \mathcal{V}$ (if it exists) such that $A = (A \cap B) \oplus C$.

Theorem

In the class of all vector spaces over a common field, and in the class of all divisible groups, if A_1, A_2, A_3 overlap, then they linearly amalgamate.

Main idea of proof.

- $A_1 \boxplus A_2 = (A_1 \ominus A_2) \oplus (A_1 \cap A_2) \oplus (A_2 \ominus A_1)$.
- Hence, no proper quotient of $A_1 \boxplus A_2$ amalgamates A_1, A_2:

$$a_1 + a_{12} + a_2 \equiv 0 \Rightarrow A_1 \ni a_1 + a_{12} \equiv -a_2 \in A_2. \qquad \square$$

Remark: three Boolean algebras can nonlinearly amalgamate.

Characterizing linear amalgamation

Theorem (informal)

Assume $\mathcal{V}$ is a "reasonable" class of structures in which every <u>two</u> overlapping structures amalgamate. Then, overlapping structures $A_1, A_2, \ldots, A_n \in \mathcal{V}$ linearly amalgamate in $\mathcal{V}$ if and only if,

Characterizing linear amalgamation

Theorem (informal)

Assume $\mathcal{V}$ is a "reasonable" class of structures in which every <u>two</u> overlapping structures amalgamate. Then, overlapping structures $A_1, A_2, \ldots, A_n \in \mathcal{V}$ linearly amalgamate in $\mathcal{V}$ if and only if, for all $m < n$ and all terms s, t generated by $A_{m+1} \cap \bigcup_{i=1}^{m} A_i$,

1. *if $s = t$ in A_{m+1}, then $s = t$ is already implied by how $A_1 \cap A_{m+1}, \ldots, A_m \cap A_{m+1}$ overlap,*

Characterizing linear amalgamation

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2. *if $s = t$ is implied by how $A_1, \ldots, A_m$ overlap, then $s = t$ is already implied by how $A_1 \cap A_{m+1}, \ldots, A_m \cap A_{m+1}$ overlap, and*

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3. *if $k \leq m$, $u \in A_k$, and $s = u$ is implied by how $A_1, \ldots, A_m$ overlap, then already $u \in A_k \cap A_{m+1}$.*

Characterizing linear amalgamation

Theorem (informal)

Assume $\mathcal{V}$ is a "reasonable" class of structures in which every two overlapping structures amalgamate. Then, overlapping structures $A_1, A_2, \ldots, A_n \in \mathcal{V}$ linearly amalgamate in $\mathcal{V}$ if and only if, for all $m < n$ and all terms s, t generated by $A_{m+1} \cap \bigcup_{i=1}^{m} A_i$,

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3. *if $k \leq m$, $u \in A_k$, and $s = u$ is implied by how $A_1, \ldots, A_m$ overlap, then already $u \in A_k \cap A_{m+1}$.*

Condition 1 says A_{m+1} does not newly equate distinct "old" terms. Conditions 2 and 3 are closure conditions on A_{m+1}. In Part II, elementary substructures, which satisfy "every" finitary closure condition, will be used to satisfy Conditions 2 and 3.

Characterizing linear amalgamation

Theorem (more formal)

Assume $\mathcal{V}$ is a class of structures such that:

- *$\mathcal{V}$ is closed with respect to isomorphism,*
- *pushouts are well-defined in $\mathcal{V}$, and*
- *every two overlapping structures in $\mathcal{V}$ amalgamate in $\mathcal{V}$.*

Then, overlapping structures $A_1, A_2, \ldots, A_n \in \mathcal{V}$ linearly amalgamate in $\mathcal{V}$ if and only if:

1. $\boxplus_{i=1}^{m} (\mathrm{id}\colon A_i \cap A_{m+1} \to A_{m+1}) : \boxplus_{i=1}^{m}(A_i \cap A_{m+1}) \to A_{m+1}$ *is injective for all $m < n$,*
2. $\boxplus_{i=1}^{m} (\mathrm{id}\colon A_i \cap A_{m+1} \to A_i) : \boxplus_{i=1}^{m}(A_i \cap A_{m+1}) \to \boxplus_{i=1}^{m} A_i$ *is injective for all $m < n$, and*
3. *for all $k \leq m < n$, the range of $\boxplus_{i=1}^{m} (\mathrm{id}\colon A_i \cap A_{m+1} \to A_i)$, intersected with the range of $\boxplus_k : A_k \to \boxplus_{i=1}^{m} A_i$, is a subset of the image $\boxplus_k[A_k \cap A_{m+1}]$.*

Open Problems

• Given $n \geq 4$, is there an interesting example of a class in which every n but not every $n+1$ overlapping structures amalgamate?

• (B.H. and H. Neumann, 1953) If finitely many overlapping finite groups are amalgamated by a group, are they amalgamated by a finite group?

• Given $n \geq 3$, is there an algorithm for deciding whether n arbitrary overlapping finite groups amalgamate?

• Investigate "weak" amalgamation of Boolean algebras where it's okay to equate distinct elements of $A_i \cup A_j$ but not okay to equate distinct elements of A_i.

(The Neumanns' 1953 paper has an example of four abelian groups that do not weakly amalgamate in the above sense.)