Higher-order amalgamation of algebraic structures, Part II

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Independence and relative completeness

Definition

- $A \leq B$ means A is a Boolean subalagebra of B.
- $A \cong B$ means A and B are isomorphic Boolean alagebras.
- If $S \subset A$, then $\langle S \rangle = \bigcap\{B \leq A \mid S \subset B\}$.
- $S \subset A$ is independent if, for all distinct $x_1, \ldots, x_{m+n} \in S$,

$$x_1 \wedge x_2 \wedge \cdots \wedge x_m \not\leq x_{m+1} \vee x_{m+2} \vee \cdots \vee x_{m+n}.$$

- A if free if $A = \langle S \rangle$ for some independent $S \subset A$.
- $A \leq_{\mathrm{rc}} B$ means $A \leq B$ and A is relatively complete, that is, for every $b \in B$, there exists $\max\{a \in A \mid a \leq b\}$.

Remark

- Up to isomorphism, free Boolean algebras are exactly the clopen algebras of generalized Cantor spaces 2^κ.
- The stone dual of a relative complete subalgebra is a continuous open surjection.

Projective Boolean algebras

Convention: Unless stated otherwise, all maps between Boolean algebras are Boolean homormorphisms and all maps between topological spaces are continuous.

Theorem

For each Boolean algebra A, the following are equivalent.

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- *The Stone space $X = \mathrm{Ult}(A)$ is an injective object, that is, for every compact Hausdorff 0-dimensional Z, closed $Y \subset Z$, and $f\colon Y \to X$, there is $g\colon Z \to X$ extending f.*

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- *A is retract of a free Boolean algebra F, that is, there is $r\colon F \to A'$ such that $A \cong A'$ and $r(a) = a$ for all $a \in A'$.*

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- *The Stone space $X = \mathrm{Ult}(A)$ is a retract of some 2^κ, that is, there is $r\colon 2^\kappa \to X'$ such that $X \cong X'$ and $r(x) = x$ for all $x \in X'$.*

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Theorem (Koppelberg)

A Boolean algebra A is projective iff it is tightly rc-filtered, that is, there is transfinite sequence $(x_\alpha \mid \alpha < \eta)$ such that $A = \langle \{x_\alpha \mid \alpha < \eta\} \rangle$ and, for all $\alpha < \eta$,

$$\langle \{x_\beta \mid \beta < \alpha\} \rangle \leq_{\mathrm{rc}} \langle \{x_\beta \mid \beta < \alpha + 1\} \rangle .$$

Another view of projective Boolean algebras

Definition

- $[A]^{<\aleph_1}$ denotes the set of all countable subsets of A.
- $\mathcal{E} \subset [A]^{<\aleph_1}$ is cofinal if for every every $S \in [A]^{<\aleph_1}$ there is some $T \in \mathcal{E}$ such that $S \subset T$.
- $\mathcal{E} \subset [A]^{<\aleph_1}$ is closed if $\bigcup \mathcal{C} \in \mathcal{E}$ for all countable chains $\mathcal{C} \subset \mathcal{E}$.

Remark. For each closed cofinal $\mathcal{E}$, there are functions $f_n \colon A^n \to A$ for $n < \omega$ such that, for every $B \in [A]^{<\aleph_1}$, if $f_n[B^n] \subset B$ for all n, then $B \in \mathcal{E}$.

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- *Among Boolean algebras of size $\aleph_2$, but not among those of size $\aleph_1$, there exists A that is not projective but is rc-filtered, that is, such that $S \leq_{\mathrm{rc}} A$ for all S in some closed cofinal $\mathcal{E} \subset [A]^{<\aleph_1}$.*

Projective Boolean algebras of size $\aleph_n$

Question (Geschke, 2002)

Suppose A is a Boolean algebra with is doubly rc-filtered, that is, such that $\langle S \cup T \rangle \leq_{\mathrm{rc}} A$ for all S, T in some closed cofinal $\mathcal{E} \subset [A]^{<\aleph_1}$. Must A be projective?

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Main Theorem (M., arXiv:1607.07944)

The answer is, "no." Moreover, for each $n < \omega$:

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Theorem

If $\mathcal{D}$ is directed, every $X \in \mathcal{D}$ is countable, and $|\bigcup \mathcal{D}| \geq \aleph_n$, then there exist $X_0, \ldots, X_{n-1} \in \mathcal{D}$ such that, for each $i < n$, $\bigcap_{j \neq i} X_j \not\subset X_i$.

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- So, I have to amalgamate non-Δ-systems.

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- If $M, N \prec \mathfrak{H}$, then $M \cap N \prec \mathfrak{H}$. (This is where I use $\sqsubset_\theta$.)

Davies sequences

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For each ordinal α, there is an $\{\alpha\}$-definable finite interval partition $(\mathcal{I}_i(\alpha) \mid i < \daleth(\alpha))$ of α such that, for every Davies sequence $(M_\beta)_{\beta<\alpha}$, $\{M_\beta \mid \beta \in \mathcal{I}_i(\alpha)\}$ is directed.

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Moreover, if $\alpha < \omega_n$, then $\daleth(\alpha) \le n$;

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Moreover, if $\alpha < \omega_n$, then $\daleth(\alpha) \leq n$; if $\alpha = |\alpha|$, then $\daleth(\alpha) \leq 1$.

High-level construction strategy

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For each ordinal α, there is an $\{\alpha\}$-definable finite interval partition $(\mathcal{I}_i(\alpha) \mid i < \daleth(\alpha))$ of α such that, for every Davies sequence $(M_\beta)_{\beta<\alpha}$, $\{M_\beta \mid \beta \in \mathcal{I}_i(\alpha)\}$ is directed.
Moreover, if $\alpha < \omega_n$, then $\daleth(\alpha) \leq n$; if $\alpha = |\alpha|$, then $\daleth(\alpha) \leq 1$.

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Requirements 1–3 ensure that $\{A_\alpha \mid \alpha < \omega_n\}$ and $\{A_\beta \mid \beta \in \mathcal{I}_i(\alpha) \cap M_\alpha\}$ will be directed.

How to guarantee amalgamation

Because any two overlapping Boolean algebras amalgamate, it is sufficient that $A_{\alpha,0}, A_{\alpha,1}, \ldots, A_{\alpha,\daleth(\alpha)-1}$ satisfy the following three conditions for linear amalgamation (from Part I of this talk):

For all $i < \daleth(\alpha)$ and all terms s, t generated by $A_{\alpha,i} \cap \bigcup_{j<i} A_{\alpha,j}$:

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Condition (a) follows from choosing $(A_\beta)_{\beta<\alpha}$ such that:

4. For all terms p, q generated by $\bigcup_{j<\daleth(\beta)} A_{\beta,j}$, if $p = q$ in A_β, then $p = q$ is already implied by how $A_{\beta,0}, \ldots, A_{\beta,\daleth(\beta)-1}$ overlap.

Boolean algebra construction strategy

• Recall that a pushout $\boxplus_{i<m} B_i$ of overlapping algebraic structures $B_0, \ldots, B_{m-1}$ is generated by a disjoint union of copies $e_i(B_i)$ of B_i and relations saying that each e_i is an isomorphism and $e_i(x) = e_j(y)$ for all $i < j < m$ and $x, y \in B_i \cap B_j$.

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- If requirements 1–4 are met for $\alpha' < \alpha$, then:
  - Each $\{A_\beta \mid \beta \in \mathcal{I}_i(\alpha) \cap M_\alpha\}$ is directed.
  - We may choose $\boxplus_{j<\daleth(\alpha)} A_{\alpha,j}$ such that $A_{\alpha,i} \leq \boxplus_{\boldsymbol{j\in s}} \boldsymbol{A_{\alpha,j}} \leq \boxplus_{j<\daleth(\alpha)} A_{\alpha,j}$ for all $i \in \boldsymbol{s} \subset \daleth(\alpha)$.

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- Condition 5 (and $n \geq 3$) ensure that A is doubly rc-filtered.
- Condition 6 ensures that A is not projective.

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• Verifying Condition 5 also requires some tricky lemmas about how $\boxplus$ and $\leq_{\mathrm{rc}}$ interact.

Some useful coherence properties

Assume η is an ordinal and, for all $\alpha < \eta$ and $i < \gimel(\alpha)$:

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3''. $A_\alpha \supsetneq A_{\alpha,i} = \bigcup\{A_\beta \mid \beta \in \mathcal{I}_i(\alpha) \cap M_\alpha\}$.

Then, for all $\alpha < \eta$, $i < j < \gimel(\alpha)$, $s \subset \gimel(\alpha)$, and $E \subset \eta$:

- If $A_\alpha \subset \bigcup_{\beta\in E} A_\beta$, then $A_\alpha \subset A_\beta$ for some $\beta \in E$.

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1. $(M_\alpha)_{\alpha<\eta}$ is a Davies sequence with $(A_\beta)_{\beta<\alpha} \in M_\alpha$.
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3″. $A_\alpha \supsetneq A_{\alpha,i} = \bigcup\{A_\beta \mid \beta \in \mathcal{I}_i(\alpha) \cap M_\alpha\}$.

Then, for all $\alpha < \eta$, $i < j < \daleth(\alpha)$, $s \subset \daleth(\alpha)$, and $E \subset \eta$:

- If $A_\alpha \subset \bigcup_{\beta \in E} A_\beta$, then $A_\alpha \subset A_\beta$ for some $\beta \in E$.
- There exists $D \subset \eta$ such that $\bigcup_{\beta \in D} A_\beta = \bigcap_{\beta \in E} A_\beta$ and $\{A_\beta \mid \beta \in D\}$ is directed.

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- $\bigcup\{A_\beta \mid \beta \in \mathcal{I}_i(\alpha)\} \in M_\alpha \cap M_\beta$ for all $\beta \in \mathcal{I}_j(\alpha)$.

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If:

A is a Boolean algebra of size $\leq \aleph_n$ and there is a closed cofinal $\mathcal{E} \subset [A]^{<\aleph_1}$ such that $\langle S_1 \cup \cdots \cup S_n \rangle \leq_{\mathrm{rc}} A$ for all $S_1, \ldots, S_n \in \mathcal{E}$,

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- By elementarity, $\left\langle \bigcup_{\beta<\alpha} A_\beta \right\rangle \leq_{\mathrm{rc}} A$.

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- Then $\langle \bigcup \{A_\beta \mid \beta \in \alpha \cap M_\alpha\} \rangle = \left\langle \bigcup_{i<\daleth(\alpha)} A_{\alpha,i} \right\rangle \leq_{\mathrm{rc}} A_\alpha$.
- By elementarity, $\left\langle \bigcup_{\beta<\alpha} A_\beta \right\rangle \leq_{\mathrm{rc}} A$.
- Haydon's theorem: A is projective iff A is of the form $\left\langle \bigcup_{\alpha<\eta} C_\alpha \right\rangle$ where
  - each C_α is countable and
  - $\left\langle \bigcup_{\beta<\alpha} C_\beta \right\rangle \leq_{\mathrm{rc}} \left\langle \bigcup_{\beta<\alpha+1} C_\beta \right\rangle$.

□

A finitary characterization of projective Boolean algebras

Theorem (M., arXiv:1607.07944)

A Boolean algebra A is projective if and only if there exists $\mathcal{D}$ such that:

- *$\mathcal{D}$ is a directed set of finite subalgebras of A,*
- *$\bigcup \mathcal{D} = A$, and*
- *for all $m < \omega$ and all $B_0, \ldots, B_{m-1} \in \mathcal{D}$,*
  for all terms s, t generated by $\bigcup_{i<m} B_i$,
  if $s = t$ in A,
  then $s = t$ is already implied by how $B_0, \ldots B_{m-1}$ overlap.

The proof uses lemmas about how $\boxplus$ and $\leq_{\mathrm{rc}}$ interact, and Koppelberg's representation of any projective A as $A = \langle \{x_\alpha \mid \alpha < \eta\} \rangle$ where

$$\langle \{x_\beta \mid \beta < \alpha\} \rangle \leq_{\mathrm{rc}} \langle \{x_\beta \mid \beta < \alpha + 1\} \rangle .$$

An open problem

Suppose that A has the ternary SFN
(where SFN = strong Freese-Nation property), that is,

- $\mathcal{D}$ is a directed set of finite Boolean algebras,
- $\bigcup \mathcal{D} = A$, and
- for all $\boldsymbol{m} \leq \mathbf{3}$ and all $B_0, \ldots, B_{m-1} \in \mathcal{D}$,
  for all terms s, t generated by $\bigcup_{i<m} B_i$,
  if $s = t$ in A,
  then $s = t$ is already implied by how $B_0, \ldots B_{m-1}$ overlap.

Must A be projective?

- I know that the following two implications are true:
  $$\text{projective} \Rightarrow \text{ternary SFN} \Rightarrow \text{doubly rc-filtered.}$$
- I know that at least one of them is strict. But which?
- I know that neither is strict for Boolean algebras of size $\leq \aleph_2$.
- I know that both of the following implications are strict:
  $$\text{projective} \Rightarrow \text{binary SFN} \Rightarrow \text{rc-filtered.}$$

Another open problem

Ščepin's examples of rc-filtered non-projective Boolean algebras include the clopen algebra of the symmetric square of 2^{ω_2}. Is there a similarly "natural" example of a doubly rc-filtered non-projective Boolean algebra?

- Main obstacle: Ščepin uses functors $F\colon \mathrm{Bool} \to \mathrm{Bool}$ for which
$$F(A) \boxplus F(B) \ncong F(A \boxplus B).$$
This is necessary for any "natural" F that breaks the doubly rc-filtered property while preserving the rc-filtered property.
- Is there a "natural" $F\colon \mathrm{Bool} \to \mathrm{Bool}$ such that, for all A, B,
$$F(A) \boxplus F(B) \cong F(A \boxplus B),$$
but for some A_0, A_1, A_2,
$$F\left(\bigboxplus_{i<3} A_i\right) \ncong \bigboxplus_{i<3} F(A_i)?$$
(Recall that $\bigboxplus_{i<3} A_i$ is not $(A_0 \boxplus A_1) \boxplus A_2$ in general.)