Unsettled linear algebra of Fourier transforms of complex-coefficient pseudo-Gaussians.

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Linear Channels

My job revolves around statistical models of random linear channels. Linear channels work like this:

- A complex tone $s(t) = e^{2\pi i f t}$ of frequency f is trasmitted.
- At position (x, y) on a receiving antenna aperture, the received signal is

$$r(t) = T(x, y, f, t)e^{2\pi i f t}.$$

- T is called the transfer function.
- More realistically, the transmitted complex signal is a continuous linear combination of tones

$$s(t) = \int_{\mathbb{R}} S(f)e^{2\pi i f t} df$$

and the total received complex signal is

$$r(t) = \iiint_{\mathbb{R}^3} A(x, y)S(f)T(x, y, f, t)e^{2\pi i f t} dx\, dy\, df$$

for some antenna aperture weight function $A(x, y)$.

Random Linear Channels

- The transfer function T is modelled as a complex-valued random process drawn from a known distribution.
- Modulo some strong assumptions, the distribution of T is characterized by an autocovariance function

$$R(\Delta x, \Delta y, \Delta f, \Delta t) = \left\langle \overline{T(x, y, f, t)} T(x + \Delta x, y + \Delta y, f + \Delta f, t + \Delta t) \right\rangle$$

- **Notation:** $\langle \bullet \rangle$ denotes mean value.
- **Notation:** $\overline{\bullet}$ denotes complex conjugation.
- When I can't get away with such strong assumptions, I have to think about higher-order moments like

$$\left\langle \overline{T(x_1, y_1, f_1, t_1,)} T(x_2, y_2, f_2, t_2,) \overline{T(x_3, y_3, f_3, t_3,)} T(x_4, y_4, f_4, t_4) \right\rangle .$$

Gaussian Functions

- Call a function $g\colon \mathbb{R}^n \to \mathbb{R}$ Gaussian if

$$g(\mathbf{x}) = g(\mathbf{0}) \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right)$$

  for some real symmetric positive-definite $n \times n$ matrix A.
- Symmetric means the transpose A^T equals A.
  There is no loss of generality is assuming A is symmetric.
- Positive-definite means $\mathbf{x}^T A\,\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

Gaussian Channel Models

- Because of the Central Limit Theorem, Gaussian autocovariance functions arise naturally.
- A simple example:

$$\begin{aligned} R_1(\Delta x, \Delta y, \ldots) &= \exp\left(-\frac{1}{2}\begin{bmatrix}\Delta x & \Delta y\end{bmatrix}\begin{bmatrix}L_x^{-2} & 0 \\ 0 & L_y^{-2}\end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix} - \cdots\right) \\ &= \exp\left(-\frac{\Delta x^2}{2L_x^2} - \frac{\Delta y^2}{2L_y^2} - \cdots\right) \end{aligned}$$

  The parameters L_x and L_y are decorrelation lengths.

- R_1 is (part of) a decent model for the autocovariance of the warping of the wavefront of a microwave transmission that just passed through the ionosphere, which has (practically) random stripes of higher and lower indices of refraction.

Pseudo-Gaussian Functions

- Call a function $g\colon \mathbb{R}^n \to \mathbb{C}$ pseudo-Gaussian if

$$g(\mathbf{x}) = g(\mathbf{0}) \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right)$$

  for some complex symmetric $n \times n$ matrix A with positive-definite real part $\Re A$.

- A pseudo-Gaussian is a Gaussian multiplied by a complex oscillating function.

$$|g(\mathbf{x})| = |g(\mathbf{0})| \exp\left(-\frac{1}{2}\mathbf{x}^T (\Re A)\,\mathbf{x}\right)$$

$$\arg(g(\mathbf{x})) = \arg(g(\mathbf{0})) - \frac{1}{2}\arg(\mathbf{x}^T (\Im A)\,\mathbf{x})$$

- Why can't we assume A is Hermitian ($A^T = \overline{A}$)? A with a non-real diagonal is an important case.

Pseudo-Gaussian Channel Models

- After that microwave passes through the ionosphere, diffraction occurs: the warps in the wavefront interfere with each other as they propagate through space or through the lower atmosphere.
- To model this diffraction, autocovariance function R is evolved from an initial value $R(z=0,\Delta x,\ldots)=R_1(\Delta x,\ldots)$ according to a differential equation like this:

$$0=\left(\frac{\partial}{\partial z}+\frac{b}{2i}\Delta f\frac{\partial^2}{\partial(\Delta x)^2}+\cdots\right)R(z,\ldots).$$

  b is some positive (real) constant.
- The final value $R(z=L,\ldots)=R_2(\ldots)$ is a pseudo-Gaussian with respect to $(\Delta x,\Delta y,\Delta t)$:

$$R_2(\Delta x,\ldots)=\frac{L_x}{\sqrt{L_x^2+ibL\Delta f}}\exp\left(-\frac{\Delta x^2}{2(L_x^2+ibL\Delta f)}-\cdots\right)$$

Fourier Transforms

- The formula for R_2 was found using Fourier transforms.
- A function $s\colon \mathbb{R}^n \to \mathbb{C}$ is Schwartz if all its partial derivatives exist everywhere and decay superpolynomially:

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^M \frac{\partial^N s(\mathbf{x})}{\partial x_{k_1} \cdots \partial x_{k_N}} = 0$$

  for all M, N, and $k_1, \ldots, k_N$.
- Pseudo-Gaussians are Schwartz.
- Define the Fourier transform $\mathcal{F}s\colon \mathbb{R}^n \to \mathbb{C}$ of s by

$$\mathcal{F}s(\mathbf{y}) = \int_{\mathbb{R}^n} \exp(-i\mathbf{y}^T\mathbf{x}) s(\mathbf{x}) d\mathbf{x}.$$

- Lemma. The Fourier transform of a Schwartz function is another Schwartz function.

Pseudo-Gaussians Transformed

- Main Lemma. If a complex symmetrix matrix A has positive-definite real part, then all eigenvalues of A have positive real parts. In particular, A is invertible.
- Theorem. The Fourier transform of pseudo-Gaussian g is another pseudo-Gaussian.
- Proof.
  1. $g(\mathbf{x}) = g(\mathbf{0}) \exp(-\mathbf{x}^T A \mathbf{x}/2)$.
  2. $\nabla g(\mathbf{x}) = -g(\mathbf{x}) A \mathbf{x}$.
  3. $i\mathcal{F}g(\mathbf{y})\mathbf{y} = -iA\nabla\mathcal{F}g(\mathbf{y})$.
  4. $-\mathcal{F}g(\mathbf{y})A^{-1}\mathbf{y} = \nabla\mathcal{F}g(\mathbf{y})$.
  5. $\mathcal{F}g(\mathbf{y}) = \mathcal{F}g(\mathbf{0}) \exp(-\mathbf{y}^T A^{-1} \mathbf{y}/2)$.
  6. Because $\mathcal{F}g$ is Schwartz, $\Re(A^{-1})$ must be positive-definite.
- Corollary. If $A^T = A$ and $\Re A$ is positive-definite, then $\Re(A^{-1})$ is also positive-definite.
- But what is $\mathcal{F}g(\mathbf{0})$?

$$\mathcal{F}g(\mathbf{0}) = g(\mathbf{0}) \int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A \mathbf{x}/2) d\mathbf{x} = \ ?$$

Pseudo-Gaussian Integrals Motivated

Again, given $A^T = A$ and $\Re A$ is positive-definite,

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A \mathbf{x}/2) d\mathbf{x} = \ ?$$

- If A is diagonal, the solution is well-known.
- If A is block diagonal with $2x2$ blocks, the solution is still not too hard.
- So far, these cases have sufficed for my channel model work.
- But mathematicians love to generalize!
- Also, future work involving higher-order moments may require the general case.

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Proof of the Main Lemma

1. Assume $A^T = A$ and $\mathbf{x}^T(\Re A)\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$.
2. Assume $\mathbf{v} \in \mathbb{C}^n$, $A\mathbf{v} = \lambda\mathbf{v}$, and $\mathbf{v}^T\overline{\mathbf{v}} = 1$.
3. Use a clever formula for $\Re\lambda$:

$$\begin{aligned} 0 &< (\Re\mathbf{v})^T(\Re A)(\Re\mathbf{v}) + (\Im\mathbf{v})^T(\Re A)(\Im\mathbf{v}) \\ &= \frac{1}{8}\sum_{+,-}(\mathbf{v} \pm \overline{\mathbf{v}})(A + \overline{A})(\overline{\mathbf{v}} \pm \mathbf{v}) \\ &= \frac{1}{4}\sum_{+,-}\Big(\lambda + \overline{\lambda} \pm \Re(\lambda\mathbf{v}^T\mathbf{v}) \pm \Re(\mathbf{v}^T\overline{A}\mathbf{v})\Big) \\ &= \Re\lambda \end{aligned}$$

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The Real Case

- Theorem. If A is an $n \times n$ real symmetric positive-definite matrix, then

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x} = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} > 0.$$

- Proof.
  1. By the spectral theorem, $A = QDQ^T$ where Q is orthogonal and D is positive diagonal.
  2. Rotating and rescaling via $x = QD^{-1/2}y$, we have

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{y}^T\mathbf{y}\right) \frac{d\mathbf{y}}{\sqrt{\det(D)}} = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}}.$$

The One-Dimensional Complex Case

- Theorem. If $\Re a > 0$, then

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2}ax^2\right) dx = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

where $\Re\sqrt{a} > 0$.

- Proof.
  1. Let $a = re^{i\phi}$ with $|\phi| < \pi/2$.
  2. The integral $\exp(-rz^2/2)dz$ along the circular arc $z = Re^{it}$ from $t = 0$ to $t = \phi/2$ goes to zero in the limit $R \to \infty$.
  3. Therefore, the Cauchy integral theorem lets us substitute $x = y/e^{i\phi/2}$ and then rotate the domain of integration back to the real line.

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2}ax^2\right) dx = \int_{\mathbb{R}} \exp\left(-\frac{1}{2}ry^2\right) \frac{dy}{e^{i\phi/2}} = \frac{\sqrt{2\pi}}{\sqrt{r}e^{i\phi/2}}.$$

Comments on Extending the Proofs

- In the real case, the rotation $\mathbb{R}^n \to \mathbb{R}^n$ via the orthogonal matrix Q preserved the integral.
- In the 1D complex case, the rotation $\mathbb{R} \to e^{i\phi/2}\mathbb{R} \subset \mathbb{C}$ preserved the integral only because $|\phi| < \pi/2$.
- Naively using $\phi + 2\pi$ in place of ϕ erroneously multiplies the integral by a factor of -1.
- For complex symmetric A, the closest thing to the spectral theorem is the Takagi factorization $A = UDU^T$ with U unitary ($\overline{U}U^T = I$) and D real nonnegative diagonal.
- But an arbitrary unitary rotation $\mathbb{R}^n \to U\mathbb{R}^n \subset \mathbb{C}^n$ will not preserve the integral.
- Main Question. Can the Takagi factorization be improved under the additional assumption that $\Re A$ is positive definite?

Complex Diagonal Examples

- Notation. If A is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite, then define $c(A)$ by

$$\frac{(2\pi)^{n/2}}{c(A)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}$$

- Theorem. If $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $\Re\lambda_k > 0$ for all k, then $c(D) = \prod_k \sqrt{\lambda_k}$ where $\Re\sqrt{\lambda_k} > 0$.
- $c(D)$ is a square root of $\det(D)$ that depends on D, not just on $\det(D)$:
  1. $\det(\mathrm{diag}(e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5})) = e^{6\pi i/5}$.
  2. $c(\mathrm{diag}(e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5})) = e^{3\pi i/5}$.
  3. $\det(\mathrm{diag}(e^{-2\pi i/5}, e^{-2\pi i/5}, 1)) = e^{-4\pi i/5} = e^{6\pi i/5}$.
  4. $c(\mathrm{diag}(e^{-2\pi i/5}, e^{-2\pi i/5}, 1)) = e^{-2\pi i/5} = -e^{3\pi i/5}$.

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Main Theorem

Assume A is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite.

- Notation.

$$\frac{(2\pi)^{n/2}}{c(A)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}$$

- Theorem. $c(A) = \prod_k \sqrt{\lambda_k}$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\Re\sqrt{\lambda_k} > 0$.
- Corollary. $c(A)^2 = \det(A)$.
- I've numerically checked that the theorem holds for 10^4 random choices for A of size $n \times n$, for each $n \leq 10$.

Proof of Main Theorem

1. The theorem holds for real symmetric matrices.
2. We can continuously deform $\Re A$ to A by the straight-line homotopy $A(t) = \Re A + t\Im A$, which preserves our hypotheses $A^T = A$ and $\Re A$ positive-definite.
3. By the main lemma, $\det(A) \neq 0$.
4. Because $c(A)^2 = \left(\prod_k \sqrt{\lambda_k}\right)^2 = \det(A) \neq 0$, if both $c(A)$ and $\prod_k \sqrt{\lambda_k}$ are continuous functions of A, then the homotopy will preserve $c(A) = \prod_k \sqrt{\lambda_k}$.
5. $A \mapsto c(A)$ is continuous because of its integral formula.
6. $\prod_k \sqrt{\lambda_k}$ is a continuous function of the unordered n-tuple $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$.
7. $z \to \sqrt{z}$ with $\Re\sqrt{z} > 0$ is continuous on $\{z \in \mathbb{C} \mid \Re z > 0\}$.
8. By the main lemma, $\Re\lambda_k > 0$.
9. The unordered n-tuple $\lambda_1, \ldots, \lambda_n$ is a continuous function of of the coefficients of $\det(\lambda I - A)$, which in turn are continuous functions of A.

Comments on the Proof

- The proof is a continuity argument.
- A more algebraic proof would yield more algebraic insight.
- A more algebraic proof might lead to an improved Takagi factorization.

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Motivation

- We know that $c(A) = \prod_k \sqrt{\lambda_k}$.
- But that is not the only way to compute $c(A)$.
- Other algorithms might turn out to be helpful for finding a more algebraic proof of the main theorem.
- At least one other algorithm is more efficient.

A Direct Computation of $c(A)$

1. Let $A = P + iS$ where P and S are real symmetric $n \times n$ matrices and P is positive definite.
2. By the spectral theorem, $P = QDQ^T$ where D is positive diagonal and Q is orthogonal.
3. Rotating and rescaling via $x = QD^{-1/2}y$, we have

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{y}^T(I + iB)\mathbf{y}\right) \frac{d\mathbf{y}}{\sqrt{\det(D)}}$$

   where $B = D^{-1/2}Q^T SQD^{-1/2}$.
4. By the spectral theorem again, $B = VEV^T$ where V is orthogonal and E is real diagonal.

A Direct Computation of $c(A)$ (continued)

5. Rotating via $y = Vz$, we have

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{y}^T(I + iB)\mathbf{y}\right)\frac{d\mathbf{y}}{\sqrt{\det(D)}} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{z}^T(I + iE)\mathbf{z}\right)\frac{d\mathbf{z}}{\sqrt{\det(D)}} = \frac{1}{\sqrt{\det(D)}}\prod_k \frac{\sqrt{2\pi}}{\sqrt{1 + i\mu_k}}$$

where $E = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and $\Re\sqrt{1 + i\mu_k} > 0$.

6. $c(A) = \sqrt{\det(D)}\prod_k \sqrt{1 + i\mu_k}$.

Thus, $c(A)^2 = \det(D)\det(I + iE) = \det(P)\det(I + iB) = \det(A)$.

A Cholesky-Type Block Factorization

Assume A is a complex symmetric matrix, that $\Re A$ is positive-definite, and that A divides into blocks as follows.

$$A = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix}$$

- If E is invertible, then

$$A = \begin{bmatrix} I & 0 \\ F^T E^{-1} & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & G - F^T E^{-1} F \end{bmatrix} \begin{bmatrix} I & E^{-1}F \\ 0 & I \end{bmatrix}$$

  and, hence, $\det(A) = \det(E)\det(G - F^T E^{-1} F)$.
- Lemma. E is invertible. Moreover, $\Re E$ and $\Re(G - F^T E^{-1} F)$ are positive-definite.

Divide and Conquer

Assume $A = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix}$ is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite.

- Notation. Given $\mathbf{b} \in \mathbb{C}^n$, define

$$\frac{(2\pi)^{n/2}}{c(A,\mathbf{b})} = \int_{\mathbb{R}^n+\mathbf{b}} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}.$$

- Lemma. $c(A,\mathbf{b}) = c(E,\mathbf{0})c(G - F^T E^{-1} F, \mathbf{0})$.
- Proof. Use induction on n and the change of variables

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \end{bmatrix} = \begin{bmatrix} \mathbf{x} + E^{-1}F\mathbf{y} \\ \mathbf{y} \end{bmatrix}$$

corresponding to the factorization

$$A = \begin{bmatrix} I & 0 \\ F^T E^{-1} & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & G - F^T E^{-1} F \end{bmatrix} \begin{bmatrix} I & E^{-1}F \\ 0 & I \end{bmatrix}.$$

Again a Square Root of the Determinant

Assume $A = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix}$ is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite.

- Notation.

$$\frac{(2\pi)^{n/2}}{c(A)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A \mathbf{x}\right) d\mathbf{x}$$

- Alternative proof of $c(A)^2 = \det(A)$:
  1. Proceed by induction on n.
  2. We already proved case $n = 1$.
  3. For $n > 1$, use $c(A) = c(E)c(G - F^T E^{-1} F)$ and $\det(A) = \det(E)\det(G - F^T E^{-1} F)$.

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Open Problems

1. Find a more algebraic proof that $c(A) = \prod_k \sqrt{\lambda_k}$ with $\Re\sqrt{\lambda_k} > 0$.
2. Find an improvement of the Takagi factorization $A = UDU^T$ of complex symmetric matrices under the assumption that $\Re A$ is positive-definite.