Amalgamating many overlapping Boolean algebras

David Milovich

Texas A&M International University

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Ternary obstructions to amalgamation

Definition. A sequence $(A_i)_{i<n}$ of Boolean algebras is overlapping if, for all i, j, the Boolean operators of A_i and A_j agree when restricted to their common domain.

Given a pair of overlapping Boolean algebras A, B, there is a Boolean algebra C extending both of them.

Moreover, an arbitrary Δ-system of overlapping Boolean algebras also has a common extension. (Koppelberg)

But three overlapping Boolean algebras (or just posets) A, B, C may not have a common extension. Minimal example:

$$x <_A y <_B < z <_C x.$$

(Generate A from x, y and the relation $x \wedge -y = 0$; similarly construct B and C.)

Some direct limits need ternary amalgamation

A set D of sets is directed if each pair $x, y \in D$ satisfies $x \cup y \subset z$ for some $z \in D$.

Proposition. If D is a directed set of countable sets and $|\bigcup D| \geq \aleph_n$, then there are $x_1, \ldots, x_n \in D$ such that $\bigcap_{j \neq i} x_j \not\subset x_i$ for all i.

Therefore, any construction of a Boolean algebra of size $\geq \aleph_3$ as a directed union of countable Boolean algebras must amalgamate non-Δ-system triples of overlapping algebras.

To ease such constructions, we combine:

1. Algebra: a sufficient condition for amalgamation.
2. Set theory: Long ω_1-approximation sequences (also known as Davies sequences).

Algebra: n-ary pushouts

Definition. A pushout of overlapping Boolean algebras $(A_i)_{i<n}$ is a Boolean algebra $\boxplus_{i<n} A_i$ generated by:

- Distinct generators $\boxplus_i(x)$ for $i < n$ and $x \in A_i \setminus \{0_{A_i}, 1_{A_i}\}$.
- Relations:
  - $\boxplus_i(x \wedge y) = \boxplus_i(x) \wedge \boxplus_i(y)$ for $x, y \in A_i$.
  - $\boxplus_i(-x) = -\boxplus_i(x)$ for $x \in A_i$.
  - $\boxplus_i(x) = \boxplus_j(x)$ if $x \in A_i \cap A_j$.

In the category of Boolean algebras and Boolean homomorphisms, $\boxplus_{i<n} A_i$, along with the morphisms $\boxplus_i \colon A_i \to \boxplus_{i<n} A_i$, is a colimit of the commutative diagram of inclusion maps $\mathrm{id}\colon \bigcap_{i\in s} A_i \to \bigcap_{i \in t} A_i$ for $\varnothing \neq t \subset s \subset n$.

Algebra: n-wise commuting subalgebras

Notation: $A \leq B$ means A is a subalgebra of B.

Definition ($n = 2$: Heindorf and Shapiro).
Given $A_i \leq B$ for $i < n$,
we say $(A_i)_{i<n}$ commutes in B if,
for every tuple of ultrafilters $U_i \in \mathrm{Ult}(A_i)$ for $i < n$,
if $U_i \cap A_j = U_j \cap A_i$ for all $i, j < n$,
then there is an ultrafilter $V \in \mathrm{Ult}(B)$ extending every U_i.

Lemma. $(A_i)_{i<n}$ commutes in B iff we can choose $\boxplus_{i<n} A_i$ such that $A_i \leq \boxplus_{i<n} A_i \leq B$ for all $i < n$.

Application: An n-ary interpolation theorem

The Interpolation Theorem of Proposition Logic.
If $\varphi \vdash \psi$, then $\varphi \vdash \chi \vdash \psi$ for some χ with all its propositional variables common to φ and ψ.

The Interpolation Theorem can be reinterpreted as a corollary of certain pairs of subalgebras of a free Boolean algebra commuting.

An n-ary generalization.
If $\bigwedge_{i<n} \varphi_i \vdash \bot$, then there exist χ_i for $i < n$ such that:

- $\varphi_i \vdash \chi_i$ for each i.
- $\bigwedge_{i<n} \chi_i \vdash \bot$.
- For each i, each propositional variable in χ_i is in φ_i and in at least one other φ_j.

Algebra: a sufficient condition for amalgamation

Notation: $\langle S \rangle$ denotes the Boolean closure of a subset S of a Boolean algebra.

Theorem 1 (M., 2016). Overlapping Boolean algebras $(A_i)_{i<n}$ mutually extend to a pushout $\bigboxplus_{i<n} A_i$ if, for all $k < m \leq n$,

1. $(A_i \cap A_m)_{i<m}$ commutes in A_m,
2. $(\boxplus_i[A_i \cap A_m])_{i<m}$ commutes in $\bigboxplus_{i<m} A_i$, and
3. $\boxplus_k[A_k \cap A_m] = \boxplus_k[A_k] \cap \left\langle \bigcup_{i<m} \boxplus_i[A_i \cap A_m] \right\rangle$ in $\bigboxplus_{i<m} A_i$.

It's not fun to verify all these conditions. Fortunately, there is a set-theoretic black box that hides these conditions behind one simpler condition.

Set theory: Long ω_1-approximation sequences

Let $\mathfrak{H}$ be the structure $(H(\theta), \in, \sqsubset_\theta)$ where:

- θ is a sufficiently large regular cardinal.
- $H(\theta)$ is the set of all sets hereditarily smaller than θ.
- $\sqsubset_\theta$ well orders of $H(\theta)$.

Definition (M., 2008).
A transfinite sequence $(M_\alpha)_{\alpha<\eta}$ is a long ω_1-approximation sequence if, for each α:

- M_α is a countable elementary substructure of $\mathfrak{H}$.
- The *sequence* $(M_\beta)_{\beta<\alpha}$ is an *element* of M_α.

Lemma. Given $(M_\alpha)_{\alpha<\eta}$ as above,

$$M_\beta \subsetneq M_\alpha \Leftrightarrow M_\beta \in M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha.$$

Warning. $\{M_\alpha \mid \alpha < \eta\}$ is not a chain if $\eta > \omega_1$.

Set theory: Coherence properties

Lemma. Given a long ω_1-approximation sequence $(M_\alpha)_{\alpha<\eta}$:

For each $\alpha < \eta$ and $B \subset \eta$,
if $M_\alpha \subset \bigcup_{\beta \in B} M_\beta$, then $M_\alpha \subset M_\beta$ for some $\beta \in B$.

For each nonempty $S \subset \eta$,
$\bigcap_{\alpha \in S} M_\alpha$ is the *directed* union of of its subsets of the form M_β.

Each $\alpha \leq \eta$ has a *finite* interval partition $I_\alpha^0, \ldots, I_\alpha^{\daleth(\alpha)-1}$ such that each $\{M_\beta \mid \beta \in I_\alpha^k\}$ is *directed*.

If $\alpha < \omega_n$, then $\daleth(\alpha) \leq n$; if α is a cardinal, then $\daleth(\alpha) = 1$.

Set theory: Pairing each M_α with a Boolean algebra

Definition. A Boolean ω_1-complex is a sequence $(A_\alpha, M_\alpha)_{\alpha<\eta}$ such that $(M_\alpha)_{\alpha<\eta}$ is a long ω_1-approximation sequence and, for all $\alpha < \eta$:

1. A_α is a Boolean algebra.
2. A_α is a subset of M_α.
3. $A_\beta \leq A_\alpha$ for all $M_\beta \in M_\alpha$.
4. $A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$.
5. $(A_\beta)_{\beta<\alpha} \in M_\alpha$.
6. $(A_\alpha^k)_{k<\daleth(\alpha)}$ commutes in A_α
   where $A_\alpha^k = \bigcup\{A_\beta \mid \beta \in I_\alpha^k \cap M_\alpha\}$.

Conditions 1–5 are trivial to satisfy provided $\vec{A}$ and $\vec{M}$ are constructed in parallel.

Condition 6 will guarantee that the sequence can be extended.

$\bigcup_{\alpha<\eta} A_\alpha$ is a directed union if η is a cardinal.

Set theory: an easier amalgamation theorem

Theorem 2 (M., 2016.) If:

- $(A_\alpha, M_\alpha)_{\alpha<\eta}$ is a Boolean ω_1-complex,
- $(M_\alpha)_{\alpha<\eta+1}$ is a long ω_1-approximation sequence, and
- $(A_\alpha)_{\alpha<\eta} \in M_\eta$,

then $B = \boxplus_{k<\beth(\eta)} A_\eta^k$ extends A_α for all $M_\alpha \in M_\eta$.

Therefore, to extend to a longer Boolean ω_1-complex $(A_\alpha, M_\alpha)_{\alpha<\eta+1}$, we may choose any A_η meeting the following requirements.

- $B \leq A_\eta$.
- A_η is a subset of M_η.
- $A_\eta \setminus \bigcup_{\alpha<\eta} M_\alpha$ is disjoint from $\bigcup_{\alpha<\eta} M_\alpha$.

Application: a higher-arity Freese-Nation property

Definition.

- Given $B \leq A$, we say B is relatively complete in A and write $B \leq_{\mathrm{rc}} A$ if for every $x \in A$ the set $\{y \in B \mid y \leq x\}$ has a maximum element.
- A Boolean algebra A has the n-ary FN if there is a club $\mathcal{C}$ of countable subalgebras of A such that $\langle B_1 \cup \cdots \cup B_{n-1} \rangle \leq_{\mathrm{rc}} A$ for all $B_1, \ldots, B_{n-1} \in \mathcal{C}$.
- A Boolean algebra A is projective if it is a retract of some free Boolean algebra F. (Retract means $A \xleftarrow{r} F \xleftarrow{e} A$; $r \circ e = \mathrm{id}$)

Theorem 3 (M., 2016).

- A is projective iff it has the n-ary FN for all n.
- If $|A| < \aleph_n$ and A has the n-ary FN, then A is projective.
- For each n, there is a Boolean algebra of size $\aleph_n$ with the n-ary FN but without the $(n+1)$-ary FN.

Application: a higher-arity strong Freese-Nation property

Definition ($n = 2$: Heindorf and Shapiro).
A Boolean A has the n-ary strong FN if it has a cofinal family $\mathcal{C}$ of *finite* subalgebras such that $B_1, \ldots, B_n$ commutes in A for $B_1, \ldots, B_n \in \mathcal{C}$.

Theorem 4 (M., 2016).

- The n-ary strong FN implies the n-ary FN.
- A is projective iff it has the n-ary strong FN for all n.
- If $|A| < \aleph_n$ and A has the n-ary strong FN, then A is projective.

Finitary applications

Let F be Stone dual of the Vietoris hyperspace functor or a nontrivial symmetric power functor.

F destroys the projectivity of the free Boolean algebra of size $\aleph_2$. (Ščepin)

Corollary (M., 2016). There is a *finite* Boolean algebra A with subalgebras B_1, B_2, B_3 that commute in A but $F(B_1), F(B_2), F(B_3)$ do not commute in $F(A)$.

The above corollary is non-constructive and gives no bound on the size of A. One of my students, René Montemayor, found that the minimal A is $\mathcal{P}(4)$.

Open problems

• To what extent do the amalgamation theorems 1 and 2 generalize to arbitrary categories? At minimum, we must assume the category has limit and colimits of all finite diagrams.

• For all $n \geq 1$, the n-ary FN does not imply the $(n+1)$-ary FN. For the strong FN, this is only known for $n = 1, 2$.
Is the 4-ary strong FN stricter stronger than the 3-ary strong FN?

• The binary strong FN is known to be strictly stronger than the binary FN. (M., 2014)
Is the ternary strong FN strictly stronger than the ternary FN?

• What is the algorithmic complexity of deciding a given list of overlapping finite Boolean algebras, reasonably encoded in N bits, has a common extension? A brute force search algorithm gives upper bounds of $\mathrm{CoNP}^{\mathrm{NP}}$ and space complexity $O(\sqrt{N})$.

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