Homeomorphism classes of hypergraph spaces

David Milovich
http://dkmj.org

Welkin Sciences
Colorado Springs, CO

Spring Topology and Dynamical Systems Conference
May 15, 2021

Table of Contents

Main Theorem

Team games

Characterizing retracts of 2^κ

Motivation

Definition

The nth symmetric power $\mathrm{SP}^n(X)$ of X is $X^n/\sim$ where $a \sim b$ iff $a = b \circ \pi$ for some bijection $\pi \colon n \to n$.

Theorem (Ščepin)

If $2 \le n < \omega \le \kappa < \omega_2$, then $\mathrm{SP}^n(2^\kappa) \cong 2^\kappa$.
But if $\kappa \ge \omega_2$ and $2 \le n < n'$, then

$$2^\kappa \not\cong \mathrm{SP}^n(2^\kappa) \not\cong \mathrm{SP}^{n'}(2^\kappa).$$

Ščepin proved this and several similar results ~40 years ago, all with ω_2 as the critical cardinal.
Are there theorems in the same spirit with critical cardinal ω_3 or higher?

Hypergraph spaces

Definition

$[S]^d$ is the set of all subsets of S with cardinality d.

Definition

The space of <u>d-regular hypergraphs</u> $X(\kappa, d)$ on κ is the power set $\mathcal{P}\big([\kappa]^d\big)$ naturally identified with the product space $2^{[\kappa]^d}$.

There's no topology here.
$X(\kappa, d) \cong 2^\kappa$ for $1 \leq d < \omega \leq \kappa$ because $|[\kappa]^d| = |\kappa|$.

More interesting hypergraph spaces

Definition

Say that $\Gamma \subset [\kappa]^d$ has no n-cliques if $[\sigma]^d \not\subset \Gamma$ for all $\sigma \in [\kappa]^n$.

(The only nontrivial case is $d < n < \kappa$.)
For example, a graph $\Gamma \subset [\kappa]^2$ is triangle-free iff it has no 3-cliques.

Definition

Let $X(\kappa, d, n)$ be the subspace of $X(\kappa, d)$ consisting of all $\Gamma \subset [\kappa]^d$ that have no n-cliques.

$X(\kappa, d, n)$ is compact Hausdorff if $d, n < \omega$.
Topologically, the only nontrivial subcase of that is

$$2 \leq d < n < \omega < \omega_1 \leq \kappa.$$

A classification problem

Definition (repeated)

$X(\kappa, d, n)$ is the space of all $\Gamma \subset [\kappa]^d$ without n-cliques (copies of $[n]^d$).

Problem

Assume $2 \leq d < n < \omega$ and $2 \leq d' < n' < \omega$.
When are $X(\kappa, d, n)$ and $X(\kappa, d', n')$ homeomorphic?

The answer is not obvious. In particular, by a Δ-system argument:

- Both $X(\kappa, d, n)$ and $X(\kappa, d', n')$ are ccc and
- both have every regular uncountable λ as a caliber.

Definition

λ is a caliber of a space if every λ-sequence of nonempty open sets has a λ-long subsequence that contains a common point.

Main Theorem

Definition (repeated)

$X(\kappa, d, n)$ is the space of all $\Gamma \subset [\kappa]^d$ without n-cliques.

Theorem

Assume $2 \le d < n < \omega$ *and* $2 \le d' < n' < \omega$.
If $\omega \le \kappa < \omega_d, \omega_{d'}$, *then*

$$2^\kappa \cong X(\kappa, d, n) \cong X(\kappa, d', n').$$

But if $\kappa \ge \omega_d$ *and* $d < d'$, *then*

$$2^\kappa \ncong X(\kappa, d, n) \ncong X(\kappa, d', n').$$

Open problem: is $X(\omega_2, 2, 3) \cong X(\omega_2, 2, 4)$?

Broader questions

If "$X(\omega_2, 2, 3) \cong X(\omega_2, 2, 4)$?" sounds interesting to you...

Definition

Given a family $\mathcal{F}$ of finite d-uniform hypergraphs, let $X(\kappa, d, \mathcal{F})$ be the set of all $\Gamma \subset [\kappa]^d$ without an induced sub-hypergraph in $\mathcal{F}$.

Question

Given $d, d', \mathcal{F}, \mathcal{F}'$, what is the least κ, if any, for which $X(\kappa, d, \mathcal{F}) \ncong X(\kappa, d', \mathcal{F}')$?

Table of Contents

Main Theorem

Team games

Characterizing retracts of 2^κ

Team games

Suppose G is a game where I and II take turns playing sequences of fixed length τ.

$$\begin{array}{lcccccc} \mathrm{I} & p_0^0, \ldots, p_{\tau-1}^0 & & p_0^1, \ldots, p_{\tau-1}^1 & & \cdots \\ \mathrm{II} & & q_0^0, \ldots, q_{\tau-1}^0 & & q_0^1, \ldots, q_{\tau-1}^1 & \cdots \end{array}$$

- Call I a team of size τ.
- Call $(p_i^0, p_i^1, p_i^2, \ldots)$ the plays of player I_i.
- Call a strategy σ for team II uncoordinated if each q_i^n played according to σ depends only on p_i^m for $m < n$.
- In other words, each player II_i following an uncoordinated strategy for II ignores I_j and II_j for $j \neq i$.

The club game for teams

- The club game $\mathrm{Club}_\tau(S, \mathcal{E})$ for finite team size τ:
  - Let S be a set S and $\mathcal{E} \subset [S]^\omega$.
  - I and II play τ-sequences of elements of S for ω rounds.

I	$(p_i^0)_{i<\tau}$		$(p_i^1)_{i<\tau}$		$\cdots$
II		$(q_i^0)_{i<\tau}$		$(q_i^1)_{i<\tau}$	$\cdots$

  - II wins iff $\bigcup_{i<\tau} \bigcup_{\alpha<\omega} \{p_i^\alpha, q_i^\alpha\} \in \mathcal{E}$.

- II has a winning strategy iff $\mathcal{E}$ contains a club subset of $[S]^\omega$.

- II has an <u>uncoordinated</u> winning strategy iff there is a club $\mathcal{C} \subset [S]^\omega$ such that $\bigcup_{i<\tau} A_i \in \mathcal{E}$ for all $A_0, \ldots, A_{\tau-1} \in \mathcal{C}$.

A game separating $X(\kappa, d, n)$ and $X(\kappa, d', n')$

Definition

- $C(X)$ is the set of all continuous $f\colon X \to \mathbb{R}$.
- For each $K \subset C(X)$, define the quotient space X/K by $a/K = b/K$ iff $f(a) = f(b)$ for all $f \in K$.
- Let $\mathcal{Q}(X)$ be the set of $K \in [C(X)]^\omega$ for which $a \mapsto a/K$ is an open map from X to X/K.
- Let $G_\tau(X) = \mathrm{Club}_\tau(C(X), \mathcal{Q}(X))$.

Theorem

Assume $2 \leq d < n < \omega$ and $2 \leq d' < n' < \omega$.

- *If $\omega \leq \kappa$ and $\tau < d'$, then II has an uncoordinated winning strategy for $G_\tau(X(\kappa, d', n'))$.*
- *If $\omega_d \leq \kappa$, then II does not have an uncoordinated winning strategy for $G_d(X(\kappa, d, n))$.*

A combinatorial reason for ω_d to be critical

Theorem (repeated)

If $2 \leq d < n < \omega$ and $\omega_d \leq \kappa$, then II does not have an uncoordinated winning strategy for $G_d(X(\kappa, d, n))$.

Definition

A set $\mathcal{I}$ is independent if $\bigcap \mathcal{A} \not\subset \bigcup \mathcal{B}$ for all disjoint finite $\mathcal{A}, \mathcal{B} \subset \mathcal{I}$.

Lemma

Assume $2 \leq d < \omega$. The following are equivalent.

- $\kappa \geq \omega_d$
- *For every club $\mathcal{E} \subset [\kappa]^\omega$, some $\mathcal{S} \in [\mathcal{E}]^d$ is independent.*

To defeat an uncoordinated strategy ζ for II in $G_d(X(\kappa, d, n))$, I use d independent "ζ-closed" subsets of $C(X)$.

An uncoordinated winning strategy outline

Theorem (repeated)

If $2 \le d' < n' < \omega \le \kappa$ and $\tau < d'$, then II has an uncoordinated winning strategy for $G_\tau(X(\kappa, d', n'))$.

- Each player II_i of team II aims for the set K_i of moves made by him and I_i to be such that $X(\kappa, d', n')/K_i$ is naturally identified with $X(C_i, d', n')$ for some $C_i \in [\kappa]^\omega$.
- To prove the quotient map $a \mapsto a/\bigcup_{i<\tau} K_i$ is open, the essential step is showing that $\tau + 1$ arbitrary finite quotients $(X(F_i, d', n'))_{i<\tau+1}$ interact only in trivial ways.
- That heart of that triviality argument is...

...a very easy lemma

Lemma

Given a cardinal r and sets A and B_i for $i < r$,
if $\varnothing \neq [A]^r \subset \bigcup_{i<r}[B_i]^r$, then there exists $i < r$ such that $A \subset B_i$.

Proof.

We prove the contrapositive.

- Suppose $v_i \in A \setminus B_i$ for each i.
- Choose $e \in [A]^r$ such that $\{v_i \mid i < r\} \subset e$.
- For each i, we have $e \notin [B_i]^r$ because $e \not\subset B_i$.
- Thus, $[A]^r \not\subset \bigcup_{i<r}[B_i]^r$. □

Table of Contents

Main Theorem

Team games

Characterizing retracts of 2^κ

Improving retractions to homeomorphisms

Theorem (repeated)

If $2 \leq d < n < \omega \leq \kappa < \omega_d$, then $X(\kappa, d, n) \cong 2^\kappa$.

To prove the above, I use Ščepin's theorem.

Definition

X is a <u>retract</u> of Y if there a continuous maps $e\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ e = \mathrm{id}_X$.

Theorem (Ščepin)

If X is a retract of some 2^λ and $\chi(p, X) = \kappa \geq \omega$ for all $p \in X$, then $X \cong 2^\kappa$.

Every closed $C \subset 2^\omega$ is a retract of 2^ω. Thus, Ščepin's theorem is the right generalization of "$P \cong 2^\omega$ for every perfect $P \subset 2^\omega$."

Retractions from uncoodinated strategies

Theorem (Ščepin)

A compact 0-dimensional space X is a retract of some 2^λ iff there is a club $\mathcal{E} \subset [C(X)]^\omega$ with $a \mapsto a/\bigcup\mathcal{S}$ is open for all $\mathcal{S} \subset \mathcal{E}$.

I proved a variant of the above with restrictions on $|\mathcal{S}|$.

Theorem

Given a compact 0-dimensional space X of weight κ, the following are equivalent.

- *X is a retract of some 2^λ.*
- *For each finite τ satisfying $\omega_\tau \leq \kappa$, II has an uncoordinated winning strategy for $G_\tau(X)$.*

Corollary

If $2 \leq d < n < \omega \leq \kappa < \omega_d$, then $X(\kappa, d, n) \cong 2^\kappa$.

Why ω_d is again critical

Theorem (repeated)

A compact 0-dimensional space X of weight κ is a retract of some 2^λ iff, each finite τ satisfying $\omega_\tau \leq \kappa$, II has an uncoordinated winning strategy for $G_\tau(X)$.

To prove that above, the following lemma is essential.

Lemma

Each ordinal α has a uniformly definable finite interval partition $\{\beta \mid \beta < \alpha\} = \bigcup\{I_i(\alpha) \mid i < \daleth(\alpha)\}$ such that:

- *The partition size $\daleth(\alpha)$ is $\leq \tau$ if $1 \leq n < \omega$ and $\alpha < \omega_\tau$.*
- *If, for each $\beta < \alpha$,*
    - *M_β is a countable elementary submodel of $H(\theta)$*
    - *and $(M_\gamma)_{\gamma<\beta} \in M_\beta$,*

  then $\bigcup\{M_\beta \mid \beta \in I_i(\alpha)\}$ is an elementary submodel of $H(\theta)$.

Characterizion theorem: proof outline

• Given X and countable elementary submodels $(M_\alpha)_{\alpha<w(X)}$, open quotient maps $X/\bigcup_{\beta<\alpha+1} M_\beta \to X/\bigcup_{\beta<\alpha} M_\beta$ are enough:

Theorem (Haydon)

Given a compact Hausdorff space X, the following are equivalent.

- *X is a retract of some 2^λ.*
- *X is the inverse limit of some continuous inverse limit system $(f_{\alpha,\beta}\colon X_\alpha \to X_\beta)_{\beta<\alpha<\kappa}$ such that:*
  - *$X_0 \subset 2^\omega$,*
  - *$X_{\alpha+1} \subset X_\alpha \times 2^\omega$, and*
  - *$f_{\alpha+1,\alpha}\colon X_{\alpha+1} \to X_\alpha$ is a continuous open surjection.*

• Using elementarity, I can reduce the problem to openness of the quotient map $X \to X/(M_\alpha \cap \bigcup_{\beta<\alpha} M_\beta)$.
• Using the lemma, I arrange for $M_\alpha \cap \bigcup_{\beta<\alpha} M_\beta$ to be a finite union of countable elementary submodels of $H(\theta)$.

Other characterizations by team games

The co-absolutes of powers of 2 can also be characterized in terms of uncoordinated winning strategies for finite teams.

I will save the details for a potential future talk.
But here is the game:

- In round $n < \omega$, team I plays open sets $(U_{n,i})_{i<\tau}$.
- Then team II plays open sets $(V_{n,i})_{i<\tau}$.
- <u>If</u> $\bigcap_i U_{n,i} \neq \varnothing$, <u>then</u> II loses if $\bigcap_i V_{n,i} = \varnothing$ or $\bigcap_i V_{n,i} \not\subset \bigcap_i U_{n,i}$.
- I wins if $\bigcup_n \bigcap_i V_i$ is dense.

This a team version of the Daniel-Kunen-Zhou open-open game.