Order properties of bases in products

David Milovich
Texas A&M International University
http://www.tamiu.edu/∼dmilovich
david.milovich@tamiu.edu

Joint work with Guit-Jan Ridderbos and Santi Spadaro

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- A preorder P is κ-**directed** if every subset smaller than κ has an (upper) bound in P.
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Convention

Order sets like κ, $[\lambda]^\kappa$, and $2^{<\kappa}$ by $\subseteq$.

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Definition

- A **local base** at p is a cofinal subset of $\tau(p, X)$.
- A π-**base** is a cofinal subset of $\tau^+(X)$.
- A **base** is a subset $\mathcal{B}$ of $\tau(X)$ that includes a local base at every point.

The **weight** $w(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a base that is of size $\leq \kappa$.	The **Noetherian type** $\mathrm{Nt}(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a base that is κ-founded.
The π-**weight** $\pi(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a π-base that is of size $\leq \kappa$.	The **Noetherian** π-**type** $\pi\mathrm{Nt}(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a π-base that is κ-founded.
The **character** $\chi(p, X)$ of p in X is the least $\kappa \geq \aleph_0$ such that p has a local base that is of size $\leq \kappa$.	The **local Noetherian type** $\chi\mathrm{Nt}(p, X)$ of p in X is the least $\kappa \geq \aleph_0$ such that p has a local base that is κ-founded.
$\chi(X) = \sup_{p \in X} \chi(p, X)$	$\chi\mathrm{Nt}(X) = \sup_{p \in X} \chi\mathrm{Nt}(p, X)$

History

- Malykhin, Peregudov, and Šapirovskiĭ studied the properties $\mathrm{Nt}(X) \leq \aleph_1$, $\pi\mathrm{Nt}(X) \leq \aleph_1$, $\mathrm{Nt}(X) = \aleph_0$, and $\pi\mathrm{Nt}(X) = \aleph_0$ in the 1970s and 1980s.
- Peregudov introduced Noetherian type and Noetherian π-type in 1997.
- Bennett and Lutzer rediscovered the property $\mathrm{Nt}(X) = \aleph_0$ in 1998.
- In 2005, Milovich introduced local Noetherian type and rediscovered Noetherian type and Noetherian π-type.

Easy upper bounds

Lemma

Every preorder P is almost $\mathrm{cf}(P)$-founded.

Corollary

For all spaces X,

- $\chi\mathrm{Nt}\,(p, X) \leq \chi(p, X)$;
- $\chi\mathrm{Nt}\,(X) \leq \chi(X)$;
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Every P is $|P|^+$-founded, so $\mathrm{Nt}\,(X) \leq w(X)^+$.

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Example

$\mathrm{Nt}\,(\beta\mathbb{N}) = w(\beta\mathbb{N})^+ = \mathfrak{c}^+$ because $\pi(\beta\mathbb{N}) = \aleph_0 < \mathrm{cf}(w(\beta\mathbb{N}))$.

Easy upper bounds for products

Theorem

If $p \in X = \prod_{i \in I} X_i$, then:

- $\mathrm{Nt}\,(X) \leq \sup_{i \in I} \mathrm{Nt}\,(X_i)$ (Peregudov, 1997)
- $\pi\mathrm{Nt}\,(X) \leq \sup_{i \in I} \pi\mathrm{Nt}\,(X_i)$
- $\chi\mathrm{Nt}\,(p, X) \leq \sup_{i \in I} \chi\mathrm{Nt}\,(p(i), X_i)$
- $\chi\mathrm{Nt}\,(X) \leq \sup_{i \in I} \chi\mathrm{Nt}\,(X)$

Large products

Theorem (essentially (Malykhin, 1981))

If $X = \prod_{\alpha<\kappa} X_\alpha$ and $|X_\alpha| > 1$ for all $\alpha < \kappa$, then

- $\kappa \geq \chi(p, X) \Rightarrow \chi\mathrm{Nt}\,(p, X) = \aleph_0$;
- $\kappa \geq \chi(X) \Rightarrow \chi\mathrm{Nt}\,(X) = \aleph_0$;
- $\kappa \geq \pi(X) \Rightarrow \pi\mathrm{Nt}\,(X) = \aleph_0$;
- $\kappa \geq w(X) \Rightarrow \mathrm{Nt}\,(X) = \aleph_0$.

Corollary

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- $\chi\mathrm{Nt}\left(X \times 2^{\chi(X)}\right) = \aleph_0$.
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Finite powers

Definition

- In a product space $X = \prod_{i \in I} X_i$, let $\mathrm{Nt}_{\mathsf{box}}(X)$ denote the least κ for which X has κ-founded base (π-base, local base at p) that consists only of boxes.
- Similarlly define $\chi\mathrm{Nt}_{\mathsf{box}}(p, X)$.
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Theorem (M.)

For all $n \in [1, \omega)$, for all spaces X:

$$\begin{aligned} \chi\mathrm{Nt}\,(p^n, X^n) = \quad \chi\mathrm{Nt}_{\mathsf{box}}(p^n, X^n) &= \chi\mathrm{Nt}\,(p, X) \\ \chi\mathrm{Nt}\,(X^n) = \quad \chi\mathrm{Nt}_{\mathsf{box}}(X^n) &= \chi\mathrm{Nt}\,(X) \\ \mathrm{Nt}_{\mathsf{box}}(X^n) &= \mathrm{Nt}\,(X) \end{aligned}$$

Could $\mathrm{Nt}(X^n) \neq \mathrm{Nt}_{\mathsf{box}}(X^n)$?

Passing to subsets

- If $\mathcal{B}$ is a local base at p in X, then $\mathcal{B}$ includes a $\chi\mathrm{Nt}(X)$-founded local base at p in X.

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Every metrizable space has a flat base.

Proof: For each $n < \omega$, pick a locally finite open cover refining the balls of radius 2^{-n}. Take the union of these covers.

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Example (M., 2009)

Set $X = \omega^\omega$. Let $\mathcal{B}$ be the set of all sets of the form $U_{s,n}$ where $s \in \omega^{<\omega}$, $n < \omega$, and $U_{s,n}$ is the set of all $f \in X$ such that $s^\frown i \subseteq f$ for some $i \leq n$. $\mathcal{B}$ a base of X, but $\mathcal{B}$ has no flat subcover.

The Square Problem

Open Question

Is $\mathrm{Nt}\left(X^2\right) \neq \mathrm{Nt}\left(X\right)$ possible? (Recall $\mathrm{Nt}\left(X\right) = \mathrm{Nt}_{\mathsf{box}}(X^2)$.)

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Partial answers (M., Spadaro)

"No," if:

- X is locally compact and metrizable;

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- X is compact, homogeneous, and has regular weight.

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- (M., 2010) Using these P and Q, we can build compact X, Y such that $\chi\mathrm{Nt}\,(X) = \chi\mathrm{Nt}\,(Y) = \aleph_1$ and $\chi\mathrm{Nt}\,(X \times Y) = \aleph_0$.

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- (Spadaro, 2010) Using a hyperspace-like construction, we can modify X and Y to get $\mathrm{Nt}(X), \mathrm{Nt}(Y) \geq \aleph_1$ and $\mathrm{Nt}(X \times Y) = \aleph_0$.

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- **Open:** Are there compact X, Y with $\mathrm{Nt}(X \times Y) < \min\{\mathrm{Nt}(X), \mathrm{Nt}(Y)\}$?

Connections with PCF theory and large cardinals

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Connections with PCF theory and large cardinals

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- **Open:** Can we have $\pi\mathrm{Nt}(X) > \aleph_1$? Equivalently, can $\langle \mathrm{Fn}(\aleph_\omega, 2, \aleph_1), \subseteq\rangle$ fail to be almost $\aleph_1$-founded?

Van Douwen's Problem

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- (M., 2007) It also follows that every known CHS has Noetherian type at most $\mathfrak{c}^+$. (Why? Not as easy. . .)

Sharp bounds

Example (Maurice, 1964)

The lexicographically ordered space $X = 2^{\omega \cdot \omega}_{\mathrm{lex}}$ is a CHS satisfying $c(X) = \mathfrak{c}$.

Example (Peregudov, 1997)

The double-arrow space X is compact, homogeneous, and $\mathrm{Nt}(X) = \mathfrak{c}^{+}$.

Does every CHS have a flat local base?

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- If we found a model of GCH with a CHS X with a local base $\mathcal{B}$ such that $\mathcal{B}$ is not almost $\aleph_1$-founded, then $c(X) > \mathfrak{c}$.

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- (Arhangel′skiĭ, 2005) If a product of linear orders is a CHS, then all factors are first countable, and hence have cellularity at most $\mathfrak{c}$.

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- Perhaps an easier question: Does GCH imply $\chi\mathrm{Nt}(X) \leq c(X)$ for all PHC X?

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Theorem (M., Ridderbos, 2007)

Given GCH, X PHC, and $\max_{p \in X} \chi(p, X) = \mathrm{cf}(\chi(X)) > d(X)$, there is a nonempty open $U \subseteq X$ such that $\chi\mathrm{Nt}(p, X) = \aleph_0$ for all $p \in U$.

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