A locally finite characterization of AE(0) and related classes of compacta

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Stone duality notation

- A *compactum* is a compact Hausdorff space.
- A *boolean space* is a compactum with a clopen base.
- Clop is the contravarient functor from boolean spaces and continuous maps to boolean algebras and homomorphisms.
  - $\mathrm{Clop}(X)$ is $(\{K \subseteq X : K \text{ clopen}\}, \cap, \cup, K \mapsto X \setminus K\})$.
  - $\mathrm{Clop}(f)(K) = f^{-1}[K]$.
- Modulo isomorphism, the inverse of Clop is the functor Ult:
  - $\mathrm{Ult}(A)$ is $\{U \subseteq A : U \text{ ultrafilter}\}$ with clopen base $\{\{U \in \mathrm{Ult}(A) : a \in U\} : a \in A\}$;
  - $\mathrm{Ult}(\phi)(U) = \phi^{-1}[U]$.

Open is dual to relatively complete.

- A boolean subalgebra A of B is called *relatively complete* if every $b \in B$ has a least upper bound in A.
  - Let $A \leq_{\mathrm{rc}} B$ abbreviate "A is relatively complete in B."
- A boolean homomorphism $\phi\colon A \to B$ is called relatively complete if $\phi[A] \leq_{\mathrm{rc}} B$.
- A boolean homomorphism ϕ is relatively complete iff $\mathrm{Ult}(\phi)$ is open.

AE(0) spaces

Definition

A boolean space X is an *absolute extensor of dimension zero*, or *AE(0)* for short, if, for every continuous $f\colon Y \to X$ with $Y \subseteq Z$ boolean, f extends to a continuous $g\colon Z \to X$.

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Given a boolean space X of weight $\leq \kappa$, the following are known to be equivalent:

- X is AE(0).
- X is Dugundji, *i.e.*, a retract of 2^κ.
- $X \times 2^\kappa \cong 2^\kappa$.
- There exists Y such that $X \cong Y \subseteq 2^\kappa$ and, for all $\alpha < \beta < \kappa$, the projection $\pi_{\alpha,\beta}\colon Y \upharpoonright \beta \to Y \upharpoonright \alpha$ is open.
- $\mathrm{Clop}(X)$ has an additive rc-skeleton, *i.e.*, if $n < \omega$, θ is a regular cardinal, and $\mathrm{Clop}(X) \in N_i \prec H(\theta)$ for all $i < n$, then $\left\langle \mathrm{Clop}(X) \cap \bigcup_{i<n} N_i \right\rangle \leq_{\mathrm{rc}} \mathrm{Clop}(X)$.

Multicommutativity

- A *poset diagram* of boolean spaces is pair of sequences $(\vec{X}, \vec{f})$ with
  - $\mathrm{dom}(\vec{X})$ a poset,
  - X_i a boolean space for all $i \in \mathrm{dom}(\vec{X})$,
  - $f_{j,i} \colon X_i \to X_j$ continuous for all $j < i$, and
  - $f_{k,i} = f_{k,j} \circ f_{j,i}$ for all $k < j < i$.
- Given a poset diagram $(\vec{X}, \vec{f})$ and $I \subseteq \mathrm{dom}(\vec{X})$, let

$$\lim(X_i : i \in I) = \left\{ p \in \prod_{i \in I} X_i : \forall \{j < i\} \subseteq I \;\; p(j) = f_{j,i}(p(i)) \right\}.$$

- Call a poset diagram $(\vec{X}, \vec{f})$ *multicommutative* if, for all $i \in \mathrm{dom}(\vec{X})$, $\prod_{j<i} f_{j,i}$ maps X_i onto $\lim(X_j : j < i)$.

A new characterization of AE(0)

- A poset P is called *locally finite* if every lower cone is finite.
- A poset diagram $(\vec{X}, \vec{f})$ is *locally finite* if $\operatorname{dom}(\vec{X})$ is locally finite **and every** X_i **is finite**.
- A poset diagram $(\vec{X}, \vec{f})$ is called a *lattice diagram* if $\operatorname{dom}(\vec{X})$ is a lattice and $\{(p, q) : f_{i\wedge j, i\vee j}(p) = f_{i\wedge j, i\vee j}(q)\}$ is the least closed transitive relation containing $\bigcup_{k\in\{i,j\}}\{(p, q) : f_{k, i\vee j}(p) = f_{k, i\vee j}(q)\}$.

Theorem (M.)

Given a boolean space X, the following are equivalent.

- *X is AE(0).*
- *X is homeomorphic to the limit of a multicommutative locally finite poset diagram.*
- *X is homeomorphic to the limit of a multicommutative locally finite lattice diagram.*

Long ω_1-approximation sequences

- For every ordinal α, let
  - $\lfloor \alpha \rfloor = \max\{\beta \leq \alpha : \beta < \omega_1 \text{ or } \exists \gamma \ \ |\alpha| \cdot \gamma = \beta\}$;
  - $\alpha = \lfloor \alpha \rfloor + [\alpha]$;

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  - $\lfloor\alpha\rfloor_n = \sum_{i<n} \lfloor[\alpha]_i\rfloor$;
  - $\urcorner(\alpha) = \min\{n < \omega : [\alpha]_n = 0\}$.
  - If $1 \leq k < \omega$ and $\alpha \leq \omega_k$, then $\urcorner(\alpha) \leq k$.
- Given θ regular and uncountable, a *long ω_1-approximation sequence* is a transfinite sequence $(M_\alpha)_{\alpha<\eta}$ of countable elementary substructures of $H(\theta)$ such that $(M_\beta)_{\beta<\alpha} \in M_\alpha$ for all $\alpha < \eta$.
- (M., 2008) If $\vec{M}$ is a long ω_1-approximation sequence and $\alpha, \beta \in \operatorname{dom}(\vec{M})$, then
  - $M_\beta \in M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha \Leftrightarrow M_\beta \subsetneq M_\alpha$;
  - for all $i < \urcorner(\alpha)$, $M_\alpha^i = \bigcup\{M_\gamma : \lfloor\alpha\rfloor_i \leq \gamma < \lfloor\alpha\rfloor_{i+1}\}$ is a directed union; hence, $M_\alpha^i \prec H(\theta)$.

n-commutativity

- Call a lattice diagram $(\vec{X}, \vec{f})$ *n-commutative* on I if, for all $i, j_0, \ldots, j_{n-1} \in I$ with $j_0, \ldots, j_{n-1} < i$, $\prod_{k \in K} f_{k,i}$ maps X_i onto $\lim(X_k : k \in \bigcup_{m<n} \{k \in \mathrm{dom}(\vec{X}) : k \leq j_m\})$.
- Call a boolean space *n-commutative* if it is homeomorphic to the limit of an locally finite lattice diagram that is n-commutative on a cofinal subset of $\mathrm{dom}(\vec{X})$.

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- The Stone dual of "2-commutative boolean space" has been studied under the name of "*strong Freese-Nation property*."
- There are 2-commutative boolean spaces of weight $\aleph_2$ that are known to not be AE(0), *e.g.*, the symmetric square of 2^{ω_2}.

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- Every locally finite poset of size $\aleph_{n-1}$ contains a cofinal suborder in which every element has at most n maximal strict lower bounds.
- Hence, a boolean space of weight $\aleph_{n-1}$ is AE(0) iff it is n-commutative.
- Hence, there are 2-commutative boolean spaces of weight $\aleph_2$ that are not 3-commutative.

The strong Freese-Nation property is strictly stronger

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If A is a boolean algebra, $(M_\alpha)_{\alpha<|A|}$ is a long ω_1-approximation sequence, and $A \in M_0$, then, for all $\alpha < |A|$, $i < \daleth(\alpha)$, and $a \in A \cap M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta$, set $\sigma_i(a) = \min\{b \in A \cap M^i_\alpha : b \geq a\}$ if it exists.

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Theorem (M., 2014)

- *A has the FN iff, for all $\vec{M}, \alpha, i, a$ as above, $\sigma_i(a)$ exists.*
- *A has the SFN only if, for all $\vec{M}$ as above, $(A, \wedge, \vee, -, \sigma_0, \sigma_1, \sigma_2, \ldots)$ is a locally finite partial algebra.*
- *There is a boolean algebra of size $\aleph_2$ that has the FN but not the SFN.*