Order-theoretic invariants in set-theoretic topology

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Van Douwen's Problem

Convention

All spaces are Hausdorff (T_2).

Definition

- The **cellularity** $c(X)$ of a space X is the least infinite upper bound of cardinalities of the pairwise disjoint family of open subsets of X.
- A space is **homogeneous** if for all $p, q \in X$, there is homeomorphism $h\colon X \to X$ such that $h(p) = q$.

Theorem (Maurice)

$2^{\omega\cdot\omega}_{\mathrm{lex}}$ is a compact homogeneous space (CHS). Moreover, it has cellularity $\mathfrak{c}$ where $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}|$.

Question (Van Douwen)

Is there a CHS with cellularity exceeding $\mathfrak{c}$?

How do you make a CHS?

Definition

- A family $\mathcal{B}$ of open neighborhoods of a point $p \in X$ is a **local base** at p if for every neighborhood U of p some $B \in \mathcal{B}$ satisfies $B \subseteq U$.
- The character $\chi(p, X)$ of p is the least infinite κ such that there is a local base of size at most κ at p.
- $\chi(X) = \sup_{p \in X} \chi(p, X)$.
- X is **first countable** if $\chi(X) = \omega$.

There is a zoo of examples of first countable CHS's (e.g., $2^{\omega \cdot \omega}_{\text{lex}}$). The other major class of CHS's is the class of compact groups (and compact loops, etc.). Any product of CHS's is a CHS.

Is there any other way to make a CHS? Van Mill found a way (via resolutions) that works (if $\mathfrak{p} > \omega_1$). I found another way (via amalgams) that works (in ZFC).

Calibers > cellularity

Definition

- A family $\mathcal{B}$ of open subsets of X is a **base** of X if for all open U and $p \in U$, some $B \in \mathcal{B}$ satisfies $p \in B \subseteq U$.
- The **weight** $w(X)$ of a space X is the least infinite κ such that X has a base of set of size at most κ.
- A regular uncountable cardinal κ is a **caliber** of a space X if for every sequence $\langle U_\alpha \rangle_{\alpha<\kappa}$ of open subsets of X, there is some $I \in [\kappa]^\kappa$ such that $\bigcap_{\alpha \in I} U_\alpha \neq \emptyset$.

Basic facts

- If κ^+ is a caliber of X, then $c(X) \leq \kappa$.
- Calibers are preserved by products and continuous images.
- If $w(X) < \kappa$, then κ is a caliber of X.
- If X is compact, then $w(X) \leq |X|$.

Why is Van Douwen's Problem hard?

Theorem (Arhangel′skiĭ and Pospišil)

If X is a CHS, then $|X| = 2^{\chi(X)}$.

Theorem (Kuz′minov)

Every compact group is **dyadic**, i.e., a continuous image of a power of 2.

(Kunen noticed that by a result of Uspenskii, this theorem generalizes to compact loops, etc.)

Van Mill's and my "exceptional" CHS's all have weight at most $\mathfrak{c}$.

Observation

Every known CHS is a continuous image of a product of compacta each with weight at most $\mathfrak{c}$. Hence, $\mathfrak{c}^+$ is a caliber of every known CHS. Hence, every known CHS has cellularity at most $\mathfrak{c}$.

Exceptional homogeneous compacta

Definition

A CHS is **exceptional** if it is not homeomorphic to a product of first countable compacta and dyadic compacta.

Let T denote the unit circle. Van Mill's exceptional CHS is built using a clever topologization of $2^\omega \times T^{\omega_1}$. (Imagine each point in 2^ω being a tiny copy of $T^{\omega_1}\ldots$) Whether this space is homogeneous is independent of ZFC.

My exceptional CHS is a quotient space of $T \times (2^{\omega\cdot\omega}_{\mathrm{lex}})^{\mathscr{S}}$.

- Let $\mathscr{S}$ denote the set of all open semicircle subsets of T.
- Given $\langle p, f\rangle, \langle q, g\rangle \in T \times (2^{\omega\cdot\omega}_{\mathrm{lex}})^{\mathscr{S}}$, declare $\langle p, f\rangle \sim \langle q, g\rangle$ if
  - $p = q$ and
  - for all $S \in \mathscr{S}$, if $p \in S$, then $f(S) = g(S)$.

How many bosses do you have?

Convention

Families of subsets of a space are ordered by inclusion.

Definition

A preordered set is κ^{op}-like if no element has κ-many greater elements.

For example, the range of a descending sequence of sets $\langle U_n \rangle_{n<\omega}$ is ω^{op}-like; the range of an ascending sequence of sets $\langle V_n \rangle_{n<\omega}$ is ω_1^{op}-like, but not ω^{op}-like.

Definition

- (Peregudov) The **Noetherian type** $\mathrm{Nt}(X)$ of a space X is the least infinite κ such that X has a κ^{op}-like base.
- The **local Noetherian type** $\chi\mathrm{Nt}(p, X)$ of $p \in X$ is the least infinite κ such that X has a κ^{op}-like local base at p.
- $\chi\mathrm{Nt}(X) = \sup_{p \in X} \chi\mathrm{Nt}(p, X)$.

The metric case

Theorem

If X is metric space, then $\mathrm{Nt}(X) = \omega$.

Proof

It suffices to build an ω^{op}-like base of X. For each $n < \omega$, let $\mathcal{U}_n$ be a locally finite refinement of the cover of X by all balls of radius 2^{-n}. Then $\bigcup_{n<\omega} \mathcal{U}_n$ is a ω^{op}-like base of X.

Question

Does ω^ω (which is $\cong \mathbb{R} \setminus \mathbb{Q}$) have a base that does not include an ω^{op}-like base? Does any space X have a base that does not include an $\mathrm{Nt}(X)^{\mathrm{op}}$-like base?

Partial Answer 1

No, if X is a σ-compact metric space.

Noetherian types and Van Douwen's Problem

Theorem A

- $\chi\mathrm{Nt}(p, X) \leq \chi(p, X)$ and $\mathrm{Nt}(X) \leq w(X)^+$ always hold.
- If X is a continuous image of a product of compacta each with weight at most λ, then $\chi\mathrm{Nt}(X) \leq \lambda$.
- If X is also homogeneous, then $\mathrm{Nt}(X) \leq \lambda^+$.

Observation

Every known CHS X satisfies $\chi\mathrm{Nt}(X) = \omega$ and $\mathrm{Nt}(X) \leq \mathfrak{c}^+$.
The double arrow space is a CHS with Noetherian type $\mathfrak{c}^+$.

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Every CHS X satisfies $\chi\mathrm{Nt}(X) \le c(X)$.

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Theorem B (GCH)

Every CHS X satisfies $\chi\mathrm{Nt}\,(X) \leq c\,(X)$.

There is (in ZFC) an inhomogeneous compactum X satisfying $\chi\mathrm{Nt}\,(X) > c\,(X) = \omega$.
(Since every known CHS satisfies $\chi\mathrm{Nt}\,(X) = \omega$, one wonders if GCH is necessary. This is an open problem.)

The Power homogeneous case

Definition

- A space X is **power homogeneous** if X^λ is homogeneous for some λ.
- The **density** $d(X)$ of a space X is the least infinite κ for which X has a dense set of size at most κ. Note that $c(X) \leq d(X)$.

Question

Is $\chi\mathrm{Nt}(X) \leq c(X)$ true of every power homogeneous compactum X? $\chi\mathrm{Nt}(X) \leq d(X)$? Does assuming GCH affect the answer?

Partial Answer (GCH) (joint with G. J. Ridderbos)

If X is a power homogeneous compactum and $\max_{p \in X} \chi(p, X) = \mathrm{cf}\, \chi(X) > d(X)$, then there is a nonempty open $U \subseteq X$ such that $\chi\mathrm{Nt}(p, X) = \omega$ for all $p \in U$.

More bases

- A family $\mathcal{B}$ of nonempty open subsets of a space X is a π-**base** if for every nonempty open $U \subseteq X$, some $B \in \mathcal{B}$ satisfies $B \subseteq U$.
- The π-**weight** $\pi(X)$ of X is the least infinite κ such that X has a π-base of size at most κ.
- The **Noetherian** π-**type** $\pi\mathrm{Nt}(X)$ of X is the least infinite κ such that X has a κ^{op}-like π-base.

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- The **Noetherian** π-**type** $\pi\mathrm{Nt}(X)$ of X is the least infinite κ such that X has a κ^{op}-like π-base.
- A family $\mathcal{B}$ of nonempty open sets is a **local** π-**base** at a point $p \in X$ if for every neighborhood U of p, some $B \in \mathcal{B}$ satisfies $B \subseteq U$.
- The π-**character** $\pi\chi(p, X)$ of p is the least infinite κ such that there is a local π-base of size at most κ at p.
- The **local Noetherian** π-**type** $\pi\chi\mathrm{Nt}(p, X)$ of a point $p \in X$ is the least infinite κ such that there is a κ^{op}-like local π-base at p.
- $\pi\chi(X) = \sup_{p \in X} \pi\chi(p, X)$; $\pi\chi\mathrm{Nt}(X) = \sup_{p \in X} \pi\chi\mathrm{Nt}(p, X)$

More connections with Van Douwen's Problem

Theorem

If X is a continuous image of a product of compacta each with weight at most λ, then $\pi \mathrm{Nt}\,(X) \leq \lambda$.

Theorem

If X is compact, then $\pi \mathrm{Nt}\,(X) \leq \chi(X)^+$.

Observation

Every known CHS X satisfies $\pi \mathrm{Nt}\,(X) \leq \omega_1$ and $\pi\chi \mathrm{Nt}\,(X) = \omega$.

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If X is compact, then $\pi\mathrm{Nt}\,(X) \leq \chi(X)^+$.

Observation

Every known CHS X satisfies $\pi\mathrm{Nt}\,(X) \leq \omega_1$ and $\pi\chi\mathrm{Nt}\,(X) = \omega$.

- $\diamondsuit$ implies there is a Suslin line that is a CHS. Every Suslin line L satisfies $\pi\mathrm{Nt}\,(L) = \omega_1$.
- It is not known if ZFC proves some CHS X satisfies $\pi\mathrm{Nt}\,(X) > \omega$.
- Worse, it is not known if any (Hausdorff) space X satisfies $\pi\chi\mathrm{Nt}\,(X) > \omega$ (in any model of ZFC).

Tukey classes

Definition (Tukey)

Given directed sets P and Q, $P \leq_T Q$ means the following equivalent conditions hold.

- For some $f\colon P \to Q$, the images of unbounded sets are unbounded.
- For some $f\colon P \to Q$, the preimages of bounded sets are bounded.
- For some $g\colon Q \to P$, the images of cofinal sets are cofinal.

Theorem

If $\mathcal{A}$ is a local base at $p \in X$, $h\colon X \to Y$ is a homeomorphism, and $\mathcal{B}$ is a local base at $h(p)$, then $\langle \mathcal{A}, \supseteq \rangle \equiv_T \langle \mathcal{B}, \supseteq \rangle$.

Theorem

If $\mathcal{A}$ is a local base at a non-isolated point $p \in X$, then $\chi\mathrm{Nt}\,(p, X) \leq \lambda$ if and only if $\langle \mathcal{A}, \supseteq \rangle \geq_T \left\langle [\chi(p, X)]^{<\lambda}, \subseteq \right\rangle$.

Tukey classes and Van Douwen's Problem

Theorem C

If X is compact and $\lambda = \min_{q \in X} \pi\chi(q, X)$, then some local base $\mathcal{B}$ in X satisfies $\langle \mathcal{B}, \supseteq \rangle \geq_T \langle [\lambda]^{<\omega}, \subseteq \rangle$.

Example

- The space $X = 2^\omega \times 2^{\omega_1}_{\text{lex}} \times 2^{\omega_2}_{\text{lex}}$ is such that $\chi(p, X) = \omega_2$ for all points p, $\pi\chi(p, X) = \omega$ for some points p, and

  $$\langle \mathcal{B}, \supseteq \rangle \equiv_T \omega \times \omega_1 \times \omega_2 \not\geq_T \left\langle [\omega_2]^{<\omega_1}, \subseteq \right\rangle$$

  for all local bases $\mathcal{B}$. Hence, $\chi\mathrm{Nt}\,(p, X) = \omega_2$ for all $p \in X$.

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  for all local bases $\mathcal{B}$. Hence, $\chi\mathrm{Nt}\,(p, X) = \omega_2$ for all $p \in X$.
- If some model of GCH has a CHS X with a local base $\mathcal{B}$ such that $\langle \mathcal{B}, \supseteq \rangle \equiv_T \omega \times \omega_1 \times \omega_2$, then $c\,(X) > \mathfrak{c}$ in this model.

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If X is compact and $\lambda = \min_{q \in X} \pi\chi(q, X)$, then some local base $\mathcal{B}$ in X satisfies $\langle \mathcal{B}, \supseteq \rangle \geq_T \langle [\lambda]^{<\omega}, \subseteq \rangle$.

Example

- The space $X = 2^\omega \times 2^{\omega_1}_{\text{lex}} \times 2^{\omega_2}_{\text{lex}}$ is such that $\chi(p, X) = \omega_2$ for all points p, $\pi\chi(p, X) = \omega$ for some points p, and

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- If some model of GCH has a CHS X with a local base $\mathcal{B}$ such that $\langle \mathcal{B}, \supseteq \rangle \equiv_T \omega \times \omega_1 \times \omega_2$, then $c\,(X) > \mathfrak{c}$ in this model.
- In every model of ZFC, we don't know if such a CHS exists, even if we replace $\omega \times \omega_1 \times \omega_2$ with $\omega \times \omega_1$ or $\omega \times \omega_2$.

Subsets of bases

Question

Can a space X have a base that does not include an $\mathrm{Nt}\,(X)^{\mathrm{op}}$-like base?

Partial Answers

1. No, if X is a σ-compact metric space.
2. No, if X is a dyadic CHS.
3. No, if X is a CHS and $w(X)$ is regular. ("$w(X)$ is regular" can be dropped if $2^{\aleph_\alpha} < \aleph_{\alpha+\omega}$ for all α.)

Answers 2 and 3 follow from the two theorems below.

- If X is compact and $\chi(p, X) = w(X)$ for all $p \in X$, then every base of X contains an $\mathrm{Nt}\,(X)^{\mathrm{op}}$-like base of X.
- If X is compact and $\pi\chi(p, X) < \mathrm{cf}\,\kappa = \kappa \leq w(X)$ for all $p \in X$, then $\mathrm{Nt}\,(X) > \kappa$.

Noetherian types of ω^*

ω^* is the space of nonprincipal ultrafilters on ω. It is compact and inhomogeneous.

Theorem (Malykhin)

MA $\Rightarrow \pi\mathrm{Nt}\,(\omega^*) = \mathfrak{c}$ and CH $\Rightarrow \mathrm{Nt}\,(\omega^*) = \mathfrak{c}$.

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Definition

- Given $R, S \subseteq \omega$, we say S **splits** R if $|R \cap S| = |R \setminus S| = \omega$.
- The **splitting number** $\mathfrak{s}$ is least size of a **splitting family**, which is a subset $\mathscr{S}$ of $[\omega]^\omega$ such that every $R \in [\omega]^\omega$ is split by some $S \in \mathscr{S}$.
- The **reaping number** $\mathfrak{r}$ is least size of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that no single $S \subseteq \omega$ splits every $R \in \mathcal{R}$.
- The **distributivity number** $\mathfrak{h}$ is the least κ such that forcing with $\langle [\omega]^\omega, \subseteq^* \rangle$ adds a new subset of κ.

Exercise: $\mathfrak{c} \geq \mathfrak{r} \geq \mathfrak{h} \geq \omega_1 \leq \mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}$.

A more precise theorem

ZFC proves each of the following statements.

- $\pi\mathrm{Nt}(\omega^*) = \mathfrak{h} \leq \mathfrak{s} \leq \mathrm{Nt}(\omega^*) \leq \mathfrak{c}^+$.
- $\chi\mathrm{Nt}(\omega^*) \leq \min\{\mathrm{Nt}(\omega^*), \mathfrak{c}\}$.
- $\pi\chi\mathrm{Nt}(\omega^*) = \omega$.
- $MA \Rightarrow \pi\mathrm{Nt}(\omega^*) = \mathfrak{c} \Rightarrow \mathrm{Nt}(\omega^*) = \mathfrak{c}$.
- $\mathfrak{r} = \mathfrak{c} \Rightarrow \mathrm{Nt}(\omega^*) \leq \mathfrak{c}$.
- $\mathfrak{r} < \mathfrak{c} \Rightarrow \mathrm{Nt}(\omega^*) \geq \mathfrak{c}$.
- $\mathfrak{r} < \mathrm{cf}\,\mathfrak{c} \Rightarrow \mathrm{Nt}(\omega^*) = \mathfrak{c}^+$.

Each of the following statements are consistent with ZFC.

- $\omega_1 = \pi\mathrm{Nt}(\omega^*) = \chi\mathrm{Nt}(\omega^*) = \mathrm{Nt}(\omega^*) < \mathfrak{c}$.
- $\omega_1 < \pi\mathrm{Nt}(\omega^*) = \chi\mathrm{Nt}(\omega^*) = \mathrm{Nt}(\omega^*) < \mathfrak{c}$.
- $\omega_1 = \pi\mathrm{Nt}(\omega^*) < \mathrm{Nt}(\omega^*) < \mathfrak{c}$.
- $\omega_1 < \pi\mathrm{Nt}(\omega^*) < \chi\mathrm{Nt}(\omega^*) = \mathfrak{c} < \mathrm{Nt}(\omega^*)$.

A combinatorial version of Noetherian type

Definition

- The **supersplitting number** $\mathfrak{ss}_2$ is the least κ such that there is a sequence $\langle S_\alpha \rangle_{\alpha<\mathfrak{c}}$ of subsets of ω such that $\{S_\alpha : \alpha \in I\}$ is a splitting family for all $I \in [\mathfrak{c}]^\kappa$.
- The (other) supersplitting number $\mathfrak{ss}_\omega$ is the least κ such that there is an $n < \omega$ and a sequence $\langle f_\alpha \rangle_{\alpha<\mathfrak{c}}$ of maps from ω to n such that for all $I \in [\mathfrak{c}]^\kappa$ and all $R \in [\omega]^\omega$, $f_\alpha \restriction R$ is not eventually constant for some $\alpha \in I$.

Theorem

$\mathrm{Nt}(\omega^*) \leq \mathfrak{ss}_\omega \leq \mathfrak{ss}_2 \leq \mathfrak{c}^+$.

Question

Is $\mathrm{Nt}(\omega^*) < \mathfrak{ss}_2$ consistent? If $\mathfrak{c}$ is regular, then $\mathrm{Nt}(\omega^*) = \mathfrak{ss}_\omega$.

Isbell's Problem

Theorem (Isbell)

There is a nonprincipcal ultrafilter $\mathcal{U}$ on ω such that $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle \mathcal{U}, \supseteq \rangle \equiv_T \left\langle [\mathfrak{c}]^{<\omega}, \subseteq \right\rangle$.

Question 1 (Isbell's Problem)

Does ZFC prove there is a nonprincipal ultrafilter $\mathcal{U}$ on ω such that $\langle \mathcal{U}, \supseteq \rangle \not\equiv_T \left\langle [\mathfrak{c}]^{<\omega}, \subseteq \right\rangle$?

Question 2

Does ZFC prove there is a nonprincipal ultrafilter $\mathcal{U}$ on ω such that $\langle \mathcal{U}, \supseteq^* \rangle \not\equiv_T \left\langle [\mathfrak{c}]^{<\omega}, \subseteq \right\rangle$?

Question 3

Does ZFC prove $\chi\mathrm{Nt}\left(\omega^*\right) > \omega$?

Theorem

$\mathrm{Yes}_3 \Rightarrow \mathrm{Yes}_2 \Leftrightarrow \mathrm{Yes}_1$.

Noetherian type and products

Theorem

- (Peredudov) $\mathrm{Nt}\left(\prod_{i\in I} X_i\right) \leq \sup_{i\in I} \mathrm{Nt}\,(X_i)$.
- (Peregudov) $\mathrm{Nt}\left(X^{w(X)}\right) = \omega$ for all spaces X.
- If $w(\prod_{i\in I} X_i) \leq |I|$ and $|X_i| \geq 2$ for all $i \in I$, then $\mathrm{Nt}\left(\prod_{i\in I} X_i\right) = \omega$.
- (Spadaro) There is a Tychonoff space Y such that $\mathrm{Nt}\,(\omega_1 \times Y) < \mathrm{Nt}\,(\omega_1) = \omega_2$.

Theorem

Suppose $\alpha < \mathfrak{c}$ and $\langle X_\beta \rangle_{\beta<\alpha}$ is a sequence of spaces each with weight at most $\mathfrak{c}$. Then $\prod_{\beta<\alpha}(\omega^* \oplus X_\beta)$ is not homeomorphic to a product of $\mathfrak{c}$-many nonsingleton spaces.

Noetherian spectra

Theorem

- $\{\mathrm{Nt}(X) : X \text{ compact}\} = \{\text{infinite cardinals}\}$.
- $\{\mathrm{Nt}(X) : X \text{ compact linear order}\} =$ $\{\text{infinite cardinals}\} \setminus (\{\omega_1\} \cup \{\text{weak inaccessibles}\})$.
- $\omega_1 \notin \{\mathrm{Nt}(X) : X \text{ compact dyadic}\} \supseteq$ $\{\omega\} \cup \{\text{singular cardinals}\} \cup \{\kappa^+ : \kappa = |\kappa| \text{ and } \operatorname{cf}\kappa > \omega\}$.

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- $\{\mathrm{Nt}\,(X) : X \text{ compact linear order}\} = \{\text{infinite cardinals}\} \setminus (\{\omega_1\} \cup \{\text{weak inaccessibles}\})$.
- $\omega_1 \notin \{\mathrm{Nt}\,(X) : X \text{ compact dyadic}\} \supseteq \{\omega\} \cup \{\text{singular cardinals}\} \cup \{\kappa^+ : \kappa = |\kappa| \text{ and } \mathrm{cf}\,\kappa > \omega\}$.
- $\mathrm{Nt}\,(\kappa + 1) = \kappa^+$ if $\kappa = \mathrm{cf}\,\kappa > \omega$.
- $\mathrm{Nt}\,(\kappa + 1) = \kappa$ if κ is a singular cardinal.
- If X is a compact linear order and $\mathrm{Nt}\,(X) \leq \kappa = \mathrm{cf}\,\kappa > \omega$, then $d(X) < \kappa$.
- Let $X = (2^\kappa \oplus 2^\lambda)/\sim$ where κ and λ are infinite cardinals and $\sim$ identifies $\langle 0 \rangle_{\alpha<\kappa}$ and $\langle 0 \rangle_{\alpha<\lambda}$. If $\kappa < \mathrm{cf}\,\lambda$, then $\mathrm{Nt}\,(X) = \lambda^+$; if $\mathrm{cf}\,\lambda \leq \kappa < \lambda$, then $\mathrm{Nt}\,(X) = \lambda$.

These are a few of my favorite proofs...

Special case of Theorem A

If X is a dyadic CHS, then $\mathrm{Nt}(X) = \omega$.

Proof ingredients

- Build an ω^{op}-like base $\mathcal{B} = \bigcup_{\alpha < w(X)} \mathcal{B}_\alpha$ by transfinite recursion of length $w(X)$.
- Compact metric spaces have especially nice ω^{op}-like bases.
- At stage α, carefully build a base $\mathcal{A}_\alpha$ of the metrizable quotient X/M_α where points are distinguished iff they are separated by a continous real-valued function in M_α, where $|M_\alpha| = \omega$ and $M_\alpha \prec H_\theta$ and θ is sufficiently large.
- $\mathcal{B}_\alpha = \{\bigcup A : A \in \mathcal{A}_\alpha\}$.

More ingredients

1. Construct $\langle M_\alpha \rangle_{\alpha < w(X)}$ such that $\langle M_\beta \rangle_{\beta < \alpha} \in M_\alpha$ for all α.
2. Use homogeneity to prove $\min_{p \in X} \pi\chi(p, X) = w(X)$. ($\pi\chi(Y) = w(Y)$ is true of all dyadic compact Y.)
3. Use (1) and (2) to choose a $\mathcal{B}_\alpha$ that has no supersets of elements of $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$.
4. Use (3) to show that for limit δ, $\bigcup_{\beta<\delta} \mathcal{B}_\beta$ is ω^{op}-like if $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$ is for all $\alpha < \delta$.
5. Deduce from (1) for each α, there exists $\alpha = \beta_0 > \cdots > \beta_n = 0$ such that for each $i < n$, $N_i = \bigcup_{\beta_i > \gamma \geq \beta_{i+1}} M_\gamma$ satisfies $M_\alpha \ni N_i \prec H_\theta$.
6. Show that each quotient map from $2^{w(X)}$ to $2^{w(X)}/N_i$ is an open map.
7. Use (5) and (6) to show that $\bigcup_{\beta<\alpha+1} \mathcal{B}_\beta$ is ω^{op}-like if $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$ is.

A forcing construction

Theorem

Let $\omega_1 \leq \operatorname{cf}\kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. Then there is a ccc forcing extension in which

$$\pi\mathrm{Nt}\,(\omega^*) = \chi\mathrm{Nt}\,(\omega^*) = \mathrm{Nt}\,(\omega^*) = \mathfrak{ss}_2 = \kappa \leq \lambda = \mathfrak{c}.$$

Proof ingredients

- Construct a κ-like, κ-directed, well-founded poset Ξ with cofinality and cardinality λ.
- Construct a (generalized) forcing iteration along Ξ; let G be a generic filter.
- At each stage $\sigma \in \Xi$, add a Cohen real C_σ, which will be Cohen generic over $V[G \restriction (\Xi \setminus \uparrow\sigma)]$.
- Since Ξ is κ-like, $\langle C_\sigma \rangle_{\sigma\in\Xi}$ witnesses $\mathfrak{ss}_2 \leq \kappa$ in $V[G]$.
- Since $|\Xi| = \lambda = \lambda^\omega$, $\langle C_\sigma \rangle_{\sigma\in\Xi}$ witnesses $\mathfrak{c} = \lambda$ in $V[G]$.

More ingredients

- Using $\mathrm{cf}(\Xi) = \lambda = \lambda^{<\kappa}$, κ-directedness of Ξ, and some bookkeeping, ensure that for each $\sigma \in \Xi$, every filter base in $V[G \restriction (\downarrow\sigma)]$ that has size less than κ has a pseudointersection in $V[G]$.
- Deduce that every filter base in $V[G]$ of size less than κ has a pseudointersection.
- Deduce that $\pi\mathrm{Nt}\,(\omega^*) \geq \kappa$ in $V[G]$.
- Extend the partial ordering of Ξ to a well ordering $\sqsubseteq$.
- Use $\sqsubseteq$ to construct an ultrafilter $\mathcal{U}$ in $V[G]$ such that every $\mathcal{V} \in [\mathcal{U}]^{<\kappa}$ has a pseudointersection in $\mathcal{U}$.
- Deduce that $\chi\mathrm{Nt}\,(\omega^*) \geq \kappa$ in $V[G]$.

How did GCH get in there?

Theorem B (GCH)

Every CHS X satisfies $\chi\mathrm{Nt}\,(X) \leq c\,(X)$.

Proof ingredients

- (Arhangel′skiĭ and Pospišil) $|Y| = 2^{\chi(Y)}$ for every CHS Y.
- (Arhangel′skiĭ) $|Y| \leq 2^{\pi\chi(Y)c(Y)}$ for every CHS Y.
- (GCH) $\chi(X) \leq \pi\chi(X)c\,(X)$
- $\chi\mathrm{Nt}\,(Z)\,\pi\chi(Z) \leq \chi(Z)$ for every space Z.
- If $\pi\chi(X) < \chi(X)$, then $\chi\mathrm{Nt}\,(X) \leq \chi(X) \leq c\,(X)$.
- So, assume $\pi\chi(X) = \chi(X)$.
- The hard part is deducing $\chi\mathrm{Nt}\,(X) = \omega$.

The hard part

- By homogeneity, we only need to show that $\chi\mathrm{Nt}(p, X) = \omega$ for some $p \in X$.
- This is equivalent to showing that $\langle \mathcal{B}, \supseteq \rangle \geq_T \langle [\chi(p, X)]^{<\omega}, \subseteq \rangle$ for some local base $\mathcal{B}$ at some $p \in X$.
- By homogeneity, $\pi\chi(p, X) = \chi(p, X) = \chi(X)$ for all $p \in X$.

Theorem C. If K is compact and $\lambda = \min_{q \in K} \pi\chi(q, K)$, then some local base $\mathcal{B}$ in K satisfies $\langle \mathcal{B}, \supseteq \rangle \geq_T \langle [\lambda]^{<\omega}, \subseteq \rangle$.

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Theorem C. If K is compact and $\lambda = \min_{q \in K} \pi\chi(q, K)$, then some local base $\mathcal{B}$ in K satisfies $\langle \mathcal{B}, \supseteq \rangle \geq_T \langle [\lambda]^{<\omega}, \subseteq \rangle$.

Proof ingredients.

- It suffices to find a point p and a sequence $\langle V_\alpha \rangle_{\alpha < \lambda}$ of neighborhoods of p such that $p \notin \mathrm{int} \bigcap_{\alpha \in I} V_\alpha$ for all $I \in [\lambda]^\omega$.
- Call a sequence $\langle \langle U_\alpha, V_\alpha \rangle \rangle_{\alpha < \zeta}$ of subsets of K **flat** if ...
- Every flat sequence of length less than λ extends to flat a sequence of length λ.
- If $\langle \langle U_\alpha, V_\alpha \rangle \rangle_{\alpha < \lambda}$ is flat, then some $p \in \bigcap_{\alpha < \lambda} \overline{U_\alpha}$ works.

Flat sequences

Call a sequence $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\zeta}$ of subsets of K **flat** if:

1. $\overline{U}_\alpha \subseteq V_\alpha$ and U_α and V_α are regular open $(\forall\alpha < \zeta)$.

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2. $\forall\alpha < \zeta\ \forall\sigma, \tau \in [\alpha]^{<\omega}\quad \bigcap_{\beta\in\sigma} U_\beta \setminus \overline{\bigcup_{\gamma\in\tau} V_\gamma}$ is empty or $\not\subseteq V_\alpha$.

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3. $\forall\sigma \in [\zeta]^{<\omega}\ \forall\langle\Gamma_i\rangle_{i<n} \in ([\zeta]^\omega)^{<\omega}\ \exists\langle\gamma_i\rangle_{i<n} \in \prod_{i<n}\Gamma_i$ $\bigcap_{\alpha\in\sigma} U_\alpha \not\subseteq \bigcup_{i<n} \overline{V}_{\gamma_i}$.

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Call a sequence $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\zeta}$ of subsets of K **flat** if:

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- Conditions (1) and (3) imply that $\bigcup_{\alpha<\zeta}\{U_\alpha, V_\alpha\}$ is centered and ω^{op}-like.
- For any finite open cover $\mathcal{W}$ of K, we can choose $U_\zeta \in \mathcal{W}$ that preserves (3). (Any V_ζ will preserve (3).)
- Therefore, there is a finite open cover that witnesses that some $p \in \bigcap_{\alpha<\lambda} \overline{U}_\alpha$ works.
- If $\zeta < \lambda$, then $\min_{q\in K} \pi\chi(q, K) \geq \lambda$ guarantees we can find $\mathcal{W}$ such that for any choice of $U_\zeta \in \mathcal{W}$, there is a V_ζ such that (2) is preserved.
- (2) guarantees that (3) is preserved at limit stages.