On cofinal types in compacta: cubes, squares, and forbidden rectangles

David Milovich
Texas A&M International University
david.milovich@tamiu.edu
http://www.tamiu.edu/∼dmilovich/

October 21, 2011
Fields Institute Set Theory Seminar

Convention

All spaces are T_3 (regular and Hausdorff).

A motivating example

A space X is **homogeneous** if for all points p, q there is a homeomorphism $h\colon X \mapsto X$ sending p to q.

(Maurice, 1964)

- Let $X = 2^{\omega^2}_{\mathrm{lex}}$ be the binary sequences of ordinal length ω^2 ordered lexicographically.
- X is compact and homogeneous.
- X has a big family of pairwise disjoint open sets:
  - For each $g \in 2^\omega$, let $U_g = \{f \in X : g000\ldots < f < g111\ldots\}$.
- More precisely, X has a **cellular family** of size $2^{\aleph_0}$.

A motivating example

A space X is **homogeneous** if for all points p, q there is a homeomorphism $h\colon X \mapsto X$ sending p to q.

(Maurice, 1964)

- Let $X = 2^{\omega^2}_{\text{lex}}$ be the binary sequences of ordinal length ω^2 ordered lexicographically.
- X is compact and homogeneous.
- X has a big family of pairwise disjoint open sets:
  - For each $g \in 2^\omega$, let $U_g = \{f \in X : g000\ldots < f < g111\ldots\}$.
- More precisely, X has a **cellular family** of size $2^{\aleph_0}$.

- We can replace ω^2 with ω^α for any countable ordinal α.
- We cannot go further: compact homogeneous linear orders cannot have increasing (or decreasing) uncountable sequences.

Van Douwen's Problem: open for over forty years

Let the **cellularity** of X, or $c(X)$, be the supremum of the cardinalities of its pairwise disjoint open families.

Is there a compact homogeneous space X with $c(X) > 2^{\aleph_0}$?

We don't know the answer in any model of set theory.

The role of connectedness

If we restrict Van Douwen's Problem to totally disconnected compacta, the problem reduces to a question about boolean algebras. There is weak evidence that nothing is lost by making this restriction:

If X is a homogeneous compactum and P is a path connected homogeneous compactum (*e.g.*, P could be a circle), then $P \times X^{|P|}$ has a quotient Q such that $c(Y) \geq c(X)$ and Q is compact, homogeneous, and path connected.

If we try adding structure to enforce homogeneity. . .

Adding first-order structure hasn't solved Van Douwen's Problem.

For example, every compact group has a (left or right) Haar probability measure, and therefore has countable cellularity.

(Hart-Kunen, 1999) If we replace "group" with "quasigroup" or various other first-order structures that enforce homogeneity, then the resulting compacta still have countable cellularity.

If we try transfinite brute force...

Can we iteratively modify a space, adding autohomeomorphisms until we're done?

For first countable totally disconnected spaces, homogeneity of the space is equivalent to homogeneity of the clopen algebra (a first-order structure). Thus, you don't have to pay attention to all ultrafilters of the clopen algebra, just all the elements.

Only in this setting has transfinite brute force built compact homogeneous spaces.

(Arhangel′skiĭ's Theorem) First countable compact spaces cannot have more than $2^{\aleph_0}$ points.

If we try large products...

Homogeneous factors

- The cellularity of a product is the supremum of the cellularity of its finite subproducts.
- All known examples of homogeneous compacta (mostly compact groups and first countable homogeneous compacta) are continuous images of products whose factors all have bases of size $\leq 2^{\aleph_0}$.
- Therefore, products of known homogeneous compacta cannot have cellularity $> 2^{\aleph_0}$.

If we try large products...

Inhomogeneous factors

- (Dow-Pearl, 1997) Infinite powers of first countable totally disconnected spaces are homogeneous.
- The hypothesis of first countability cannot be removed, *e.g.*, no power of $\omega_1 + 1$ is homogeneous.
- (Many authors) Many theorems about the class of homogeneous spaces (*e.g.*, $|X| \leq 2^{\pi\chi(X)c(X)}$) have been proven true of the power homogeneous spaces (*i.e.*, those spaces X for which some X^κ is homogeneous).
- (Kunen, 1990) Products of infinite compact F-spaces (*e.g.*, $\beta\omega \setminus \omega$) are not homogeneous.
- (Arhangel′skiĭ, 2005) A product of compact linear orders is not homogeneous unless all factors are first countable.

Enter order theory

Rectangular local bases imply large cellularity.

- **Convention:** Subsets of spaces are ordered by $\supseteq$.
- If a point in a space has a local base $\mathcal{B}$ or order type $\omega \times \omega_1 \times \omega_2$, then that space has a cellular family of size $\aleph_2$.
- All points in $X = 2^{\omega}_{\mathrm{lex}} \times 2^{\omega_1}_{\mathrm{lex}} \times 2^{\omega_2}_{\mathrm{lex}}$ have local bases of order type $\omega \times \omega_1 \times \omega_2$.
- X is compact but not, alas, homogeneous. (Some, but not all, points have countable local π-bases.)

Enter order theory

Rectangular local bases imply large cellularity.

- **Convention:** Subsets of spaces are ordered by $\supseteq$.
- If a point in a space has a local base $\mathcal{B}$ or order type $\omega \times \omega_1 \times \omega_2$, then that space has a cellular family of size $\aleph_2$.
- All points in $X = 2^{\omega}_{\mathrm{lex}} \times 2^{\omega_1}_{\mathrm{lex}} \times 2^{\omega_2}_{\mathrm{lex}}$ have local bases of order type $\omega \times \omega_1 \times \omega_2$.
- X is compact but not, alas, homogeneous. (Some, but not all, points have countable local π-bases.)
- However, we haven't proved that no homogeneous compacta can have a local base with order type $\omega \times \omega_1 \times \omega_2$.
- Also, proving that would be very interesting in itself.

Cofinal types vs. order types.

Two preorders P, Q are **cofinally equivalent** (written $P \equiv_{\mathrm{cf}} Q$) if there is a preorder R with cofinal subsets P', Q' order-isomorphic to P, Q (respectively).

E.g., $\mathbb{Q} \equiv_{\mathrm{cf}} \{\sqrt{n} : n \in \mathbb{N}\}$ because both are cofinal in $\mathbb{R}$.

Less trivially, $P = \omega \times \omega_1$ (with the product order $x \leq y \Leftrightarrow x_0 \leq y_0 \wedge x_1 \leq y_1$) is cofinally equivalent to $Q = (\omega \times \omega_1, \trianglelefteq)$ where $x \vartriangleleft y \Leftrightarrow x_0 < y_0 \wedge x_1 < y_1$, even though P has uncountable chains and no infinite antichains, while Q has uncountable antichains and no uncountable chains.

(For directed sets (*e.g.*, local bases), cofinal equivalence $\Leftrightarrow$ Tukey equivalence.)

Advantages of cofinal types

In general, two local bases at the same point may have different order types, but they always have the same cofinal type.

Instead of considering order types of particular local bases at $p \in X$, we only consider the cofinal type of $\mathrm{Nbhd}(p, X)$, the set of all neighborhoods of p in X.

The cofinal type of $\mathrm{Nbhd}(p, X)$ does not change if switch the ordering from $\supseteq$ to $\sqsupseteq$ where $U \sqsupset V$ means the interior of U contains the closure of V.

Rectangular cofinal types and cellularity

In any space X, if $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$, then $c(X) \geq \aleph_2$.

More generally:

- Definition. $\mathrm{cf}(P)$ is the least of the sizes of cofinal subsets of P.
- Definition. $\mathrm{add}(P)$ is the least of the sizes of unbounded subsets of P.
- Theorem. If $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} D \times E$ and $\mathrm{cf}(D) < \mathrm{add}(E) < \infty$, then X has a cellular family of size $\mathrm{add}(E)$.

Proof sketch.

- The set $\mathrm{RO}(p, X)$ of regular open neighborhoods is a cofinal subset of $\mathrm{Nbhd}(p, X)$.
- In the suborder $\mathrm{RO}(p, X)$, every nonempty set has a greatest lower bound.
- Hence, there is a monotone map $f\colon D \times E \to \mathrm{RO}(p, X)$ with cofinal range.
- Because $\mathrm{cf}(\mathrm{RO}(p, X)) = \mathrm{cf}(D \times E) > \mathrm{cf}(D)$, $f(d, \bullet)$ cannot stabilize for all $d \in D$.
- Hence, we get a chain $\vec{U}$ of length $\mathrm{add}(E)$ in $\mathrm{RO}(p, X)$.
- To get a cellular family, take differences $U_i \setminus \overline{U_{i+1}}$.

Are rectangles are compatible with homogeneity?

Does any point p in any infinite homogeneous compactum X satisfy...

- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$? My conjecture is "no."

Are rectangles are compatible with homogeneity?

Does any point p in any infinite homogeneous compactum X satisfy...

- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1$? My conjecture is "no."

Are rectangles are compatible with homogeneity?

Does any point p in any infinite homogeneous compactum X satisfy...

- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega_1 \times \omega_2$? No, because every countably infinite $A \subseteq X$ has a limit point in X.

Are rectangles are compatible with homogeneity?

Does any point p in any infinite homogeneous compactum X satisfy...

- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega_1 \times \omega_2$? No, because every countably infinite $A \subseteq X$ has a limit point in X.
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_2$? No; see the next slide.

Are rectangles are compatible with homogeneity?

Does any point p in any infinite homogeneous compactum X satisfy...

- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1$? My conjecture is "no."
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega_1 \times \omega_2$? No, because every countably infinite $A \subseteq X$ has a limit point in X.
- $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_2$? No; see the next slide.

In fact, in every known homogeneous compactum X, every $p \in X$ satisfies $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} ([\kappa]^{<\omega}, \subseteq)$ where $\kappa = \mathrm{cf}(\mathrm{Nbhd}(p, X))$. If κ is uncountable, then $[\kappa]^{<\omega}$ is not cofinally equivalent to any finite product of ordinals.

Skinny rectangles

- If X is compact, then not all $p \in X$ satisfy $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_2$.
- More generally, if κ is a regular cardinal and $\mathrm{cf}(D) < \kappa < \mathrm{add}\, E < \infty$, then not all neighborhood filters in a compactum can be cofinally equivalent to $D \times E$.
- (This rules out $\omega \times \omega_1 \times \omega_3$, $\omega \times \omega_2 \times \omega_3$, etc.)

Proof sketch.

- As before, we get a chain of length $\operatorname{add} E$ of regular open neighborhoods of a point p, assuming $\operatorname{Nbhd}(p, X) \equiv_{\mathrm{cf}} D \times E$.
- From this we build an $\operatorname{add} E$-long free sequence $\vec{x}$ of points.
- Choose q in the closure of $\vec{y} = \vec{x} \upharpoonright \kappa$ but in the exterior of all initial segments of y.
- The exteriors of these initial segments form an unbounded chain of length κ in $\mathrm{RO}(q, X)$.
- No preorder $P \equiv_{\mathrm{cf}} D \times E$ has an unbounded chain of length κ. Contradiction!

π-character

- A local π-base at a point p is a family $\mathcal{F}$ of nonempty open sets such that every neigbhorhood of p contains an element of $\mathcal{F}$.
- The π-character $\pi\chi(p, X)$ is the least of the sizes of local π-bases at p.
- The character $\chi(p, X)$ is the least of the sizes of local bases at p, *i.e.*, $\chi(p, X)$ is the cofinality of $\mathrm{Nbhd}(p, X)$.
- $\pi\chi(X) = \sup_{\in X} \pi\chi(p, X)$ and $\chi(X) = \sup_{\in X} \chi(p, X)$
- It is not known if a homgeneous compactum X can satisfy $2^{\pi\chi(X)} < 2^{\chi(X)}$.
- If X is compact $\min_{p\in X} \pi\chi(p, X) \geq \omega_1$, then at least one $\mathrm{Nbhd}(p, X)$ has an uncountable subset that is strongly unbounded (*i.e.*, all infinite sets are unbounded), implying $\mathrm{Nbhd}(p, X)$ is not cofinally equivalent to any $D \times E$ where $\mathrm{cf}(D) < \mathrm{add}(E) < \infty$.

Another connection to cellularity

If X is a homogeneous compactum and GCH holds, then we can say more: if $p \in X$ and every cofinal subset of $\mathrm{Nbhd}(p, X)$ has a bounded set of size κ, then $c(X) \geq \kappa^+$.
Note that if $\mathrm{cf}(D) \leq \kappa < \mathrm{add}(E) < \infty$, then every cofinal subset of $D \times E$ has a bounded subset of size κ.

Some other "rectangular" results.

- If $\mathcal{F}$ and $\mathcal{G}$ are nonprincipal filters on ω and we order each of them by $\supseteq$, then the Fubini product $\mathcal{F} \otimes \mathcal{G}$ is cofinally equivalent to the product order $\mathcal{F} \times \mathcal{G}^\omega$.
- Corollary. $\mathcal{F} \otimes \mathcal{G} \otimes \mathcal{G} \equiv_{\mathrm{cf}} \mathcal{F} \otimes \mathcal{G}$.
- Corollary. If $\mathcal{F}$ and $\mathcal{G}$ are P-filters, then $\mathcal{F} \otimes \mathcal{G} \equiv_{\mathrm{cf}} \mathcal{G} \otimes \mathcal{F}$.